

## DONALDSON WALL-CROSSING FORMULAS VIA TOPOLOGY

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ABSTRACT. The wall-crossing formula for Donaldson invariants of simply connected four manifolds with  $b^+ = 1$  is shown to be a topological invariant of the manifold, for reducibles with two or fewer singular points. The explicit formulas derived agree with those of [FQ] and [EG].

## 0: Definitions and Notation

Although the Seiberg-Witten invariants have replaced Donaldson invariants as the tool of choice for answering questions about the differential topology of four manifolds, the Donaldson invariants of simply connected, smooth four manifolds with  $b^+ = 1$ , as defined in [Ko],[KoM], still pose interesting questions. These manifolds are the only known examples of manifolds whose Donaldson invariants are not of simple type (see [KoL]). Unlike manifolds with  $b^+ > 1$ , the Donaldson invariants of manifolds with  $b^+ = 1$  are not metric independent, but vary with the metric according to a “wall-crossing formula”. This wall-crossing formula is of interest in its own right, especially in comparing the Donaldson and Seiberg-Witten moduli spaces.

Let  $P \rightarrow X$  be an  $SO(3)$  principal bundle with  $p_1(P) = p$  over a smooth, simply connected four manifold  $X$ ,  $b_+^2(X) = 1$ , and  $c$  an integer lift of  $w_2(P)$ .  $\overline{\mathcal{M}}(P_p, g_X)$  will denote the Uhlenbeck compactification of the moduli space of connections on  $P$  which are anti-self-dual with respect to the metric  $g_X$ . If there are no reducible connections in  $\mathcal{M}(P_p, g_X)$ , then the map  $\mu : H_2(X; \mathbb{Z}) \rightarrow H^2(\mathcal{M}(P_p, g_X); \mathbb{Q})$  is defined by taking the slant product of the Pontrjagin class of the universal bundle  $\mathbb{P} \rightarrow \mathcal{M}(P_p, g_X) \times X$ ,  $\mu(x) = -\frac{1}{4}p_1(\mathbb{P})/x$ . If there are reducibles, the space giving the universal bundle is no longer a bundle. It is shown in [DK, Thm 5.1.21] that the  $\mu(x)$  class can be non-trivial on the link of a reducible and thus not be extendable across the reducible. For a generic metric  $g_X$ , there will be no reducibles and the cohomology class  $\mu(x)$  extends to  $\overline{\mu}(x) \in H^2(\overline{\mathcal{M}}(P_p, g_X); \mathbb{Q})$  as described in [FM]. One would like to define Donaldson invariants by pairing these classes  $\overline{\mu}(x)$  with the homology class given by the compactified moduli space. This pairing will not be independent of the metric: if one tries to use the moduli space associated to a path of metrics  $g_t$  to form a cobordism between the moduli spaces of the endpoint metrics,

$$\overline{\mathcal{M}}(P_p, g_\lambda) = \{([A], t) : [A] \in \overline{\mathcal{M}}(P_p, g_t)\}$$

there may be reducibles in this space, so that the  $\overline{\mu}(x)$  class would not be defined on it.

A reducible connection gives a reduction of  $P$  to an  $S^1$  bundle  $Q_\alpha$ ,  $c_1(Q_\alpha) = \alpha$ ,  $P = Q_\alpha \times_{S^1} SO(3)$ . The curvature of the reducible connection must then be an ASD harmonic two form, thus perpendicular to the

ray of self-dual two forms  $\omega(g_X)$  which are harmonic with respect to the metric  $g$ . There will be a reducible connection  $[\alpha]$  corresponding to  $\pm\alpha$  in  $\overline{\mathcal{M}}(P_p, g_X)$  only if

$$\omega(g_X) \in W^\alpha = \{x \in H^2(X; \mathbb{R}) : x^2 > 0, x \cdot \alpha = 0\}$$

where  $H^2(X; \mathbb{R})$  is identified with the space of harmonic forms. We call  $W^\alpha$  the wall associated to  $\alpha$ .

A reducible connection  $[\alpha]$  need not be in the highest stratum of  $\overline{\mathcal{M}}(P_p, g_X)$ . If  $p = p_1(P)$ , and we write  $r = (\alpha^2 - p)/4$ , then the reducible  $[\alpha]$  will lie in the strata of  $\overline{\mathcal{M}}(P_p, g_X)$  contained in  $\mathcal{M}(P', g) \times \Sigma^r(X)$ , where  $p_1(P') = p + 4r$  and  $w_2(P') = w_2(P)$ . The set of walls  $W^\alpha$  where  $\alpha \bmod 2 = w_2(P)$  and  $\alpha^2 > p$  is called the set of  $P$ -walls. The set of chambers  $\mathcal{C}$  of  $X$  is the set of components of the complement, in the positive cone of  $H^2(X; \mathbb{R})$ , of the  $P$ -walls. The Donaldson invariant associated to the bundle  $P$  will change only when the period point of the metric passes through a  $P$ -wall. We write  $\tilde{\mathcal{C}}$  for the pre-image of  $\mathcal{C}$  in the space of metrics under the period map, so the Donaldson invariant depends only on the element of  $\tilde{\mathcal{C}}$  in which the metric lives. We can then define the Donaldson invariants

$$D_{d,c}^X : \tilde{\mathcal{C}} \rightarrow \text{Sym}^d(H_2(X; \mathbb{Z}) \oplus H_0(X; \mathbb{Z}))$$

to be

$$D_{d,c}^X(\tilde{C})(z) = \langle \overline{\mu}(z), [\overline{\mathcal{M}}(P_p, g_X)] \rangle .$$

Here  $g_X$  is a metric such that  $g \in \tilde{C}$ ,  $d = -p + 3$ . The integer lift of  $w_2(P)$ ,  $c$  and a choice of an orientation of  $H_0(X) \otimes H_+^2(X)$  give an orientation to the moduli space.

A reducible anti-self dual connection  $[\alpha]$  corresponding to the line bundle  $Q_\alpha$  with  $c_1(Q_\alpha) = \alpha$  where  $\alpha^2 - p = 4r$  gives a family of reducibles:

$$[\alpha] \times \Sigma^r(X)$$

in the compactification. Let  $D(\alpha, g_0)$  denote the link of this family of reducibles in the parametrized moduli space  $\overline{\mathcal{M}}(P_p, g_\lambda)$ , then we define the difference term  $\delta_P(\alpha, g_0)$  by

$$\delta_P(\alpha, g_0)(z) = \langle \overline{\mu}(z), [D(\alpha, g_0)] \rangle .$$

The change in the Donaldson invariant along a path of metrics  $g_t$  is given by

$$D_{d,c}^X(g_1) - D_{d,c}^X(g_{-1}) = \sum_{\alpha} \epsilon(c, \alpha) \delta(\alpha, g_\alpha)$$

where  $g_\alpha$  is the metric in the path  $g_t$  where the reducible  $\alpha$  occurs and  $\epsilon(c, \alpha) = (-1)^{((c-\alpha)^2/2)}$ . This sum is over all reducibles that appear in the path of metrics.

In [Kom], an argument is given that the difference term  $\delta_P(\alpha, g)$  is independent of the metric  $g$  and depends only on the reducible. This is necessary to show that the Donaldson invariant depends not on the metric component  $\tilde{C}$  but only the chamber  $C$ . There are some technical difficulties with the argument given there (these difficulties are outlined in Section 3.4). Although the result is almost certainly true, we do not use it in this calculation.

The correction terms  $\delta_P(\alpha, g)$  remain mysterious, because of the difficulty in describing neighborhoods of a reducible. The lower the stratum in which the reducible appears, the harder the calculation appears to be. That is, if  $\alpha^2 - p_1 = 4r$ , then  $r$ , the number of points of totally concentrated curvature, measures the complexity. For  $r = 0$ , the difference term has been calculated in [Do,Ko], for  $r = 1$  by Yang in [Y] for  $SU(2)$  bundles. Calculations for larger  $r$  in the case of algebraic manifolds have been done in [EG] and [FQ].

In this paper, an expression for reducibles with  $r = 1, 2$  is obtained, matching those in [FR] and [EG]. The Kotschick-Morgan conjecture states that

$$\delta_P(\alpha) = \sum_{i=0}^r a_i(r, d, X) q^{r-i} \alpha^{d-2r-2i}$$

where  $q$  is the intersection form of  $X$  and the co-efficients  $a_i$  depend only on  $r, d$  and the homotopy type of  $X$ . Assuming this conjecture, Gottische, [G], has derived formulas for the coefficients  $a_i$  in terms of modular forms.

The formulas of Gottische also underlie the relation of the Seiberg-Witten and Donaldson invariants. Pidstrigach and Tyurin, in [PT], propose a program to compare the two (see [BGP] for a list of references of similar ideas). They define a larger moduli space which gives a cobordism between a link of the anti-self dual moduli space and the link of singularities of the form  $M_L^{SW} \times \Sigma^r(X)$ , where  $M_L^{SW}$  is a Seiberg-Witten moduli space. The link of  $M_L^{SW} \times \Sigma^r(X)$  has a description similar to but more complicated than  $D(\alpha, g)$ . Pairing certain cohomology classes with the link of the anti-self dual moduli space gives a multiple of the Donaldson polynomial. If we could calculate the pairing of these cohomology classes with the link of the  $M_L^{SW} \times \Sigma^r(X)$ , we could give an explicit formula relating the Seiberg-Witten and Donaldson invariants. To understand the link of the singularities  $M_L^{SW} \times \Sigma^r(X)$ , it would then be helpful to understand the “easy” case of  $D(\alpha, g)$ .

This paper is organized as follows. In section one, we give background information on an open neighborhood of  $[\alpha] \times \Sigma^r(X)$  and the cohomology classes. In section two, we perform the  $r = 1$  calculation. The heart of this paper lies in section three, the  $r = 2$  calculation, which is divided into four parts. Taubes’ gluing data description of a neighborhood of  $[\alpha] \times \Sigma^2(X)$  requires two open sets, one for the off-diagonal points and one for the diagonal points. In sections 3.1 and 3.2, we give compactifications of these open sets and compute pairings. The pairing with the open set of points on the diagonal, requires additional work, done in section 3.3, to describe the moduli space of charge two instantons on  $S^4$ . The final section, 3.4, computes the correction term where the two compactifications are compared.

## 1: Gluing Data Bundles and Links of Reducibles

We consider a family of reducibles,  $[\alpha] \times \Sigma^r(X)$ , where  $[\alpha]$  is a reducible anti-self dual connection arising from the reduction  $Q_\alpha \times_{S^1} SO(3)$ , where  $Q_\alpha$  is an  $S^1$  bundle with  $c_1(Q_\alpha) = \alpha$  and  $r = (\alpha^2 - p_1)/4$ . The space  $[\alpha] \times \Sigma^r(X)$  is a stratified set, the strata,  $\Sigma$ , of  $\Sigma^r X$  are given by partitions of  $r$ . If  $N = -\alpha^2 - 2$ , then a neighborhood of the background connection in the parametrized moduli space  $\mathcal{M}(P_{p+4r}, g_\lambda)$  is given by  $\mathbb{C}^N/\Gamma_\alpha$ . There is a description due to Taubes, [T], [FM] of a neighborhood of  $\mathbb{C}^N/\Gamma_\alpha \times (\Sigma \setminus \nu(\Delta))$  where  $\nu(\Delta)$  is a neighborhood of the big diagonal (i.e. a neighborhood of  $\overline{\Sigma} \setminus \Sigma$  in  $\Sigma$ ).

Let  $Z_k(\epsilon)$  be the Uhlenbeck compactification of the space of centered, charge  $k$  instantons on  $S^4$ . That is, if  $x : S^4 \setminus s \rightarrow \mathbb{R}^4$  is stereographic projection from the south pole,

$$Z_{n_i}(\epsilon) = \{[A] \in \overline{\mathcal{M}}(P_{n_i}, g_{S^4}) : \int x_i |F_A|^2 dx = 0, \int |x|^2 |F_A|^2 dx < \epsilon\}.$$

Here,  $|F_A| dx$  includes the points of Dirac measure for the generalized connection  $[A]$ . Because the connections in  $Z_k(\epsilon)$  are concentrated near the north pole, we can extend the space of connections framed at the south pole over this set to get  $Z_k^s(\epsilon)$ . There is an  $SO(4)$  action on the space coming from lifting the  $SO(4)$  rotation on  $S^4$  to the principal bundle and an  $SO(3)$  action on the south pole frame.

If the stratum  $\Sigma$  is described by  $(n_1, \dots, n_l)$  where  $\sum n_i = r$ , then a neighborhood of  $\mathbb{C}^N/\Gamma_\alpha \times (\Sigma \setminus \nu(\Delta))$  is described as follows. Let  $Q = Q_\alpha$  be the  $S^1$  reduction corresponding to  $\alpha$ ,  $Fr(X)$  be the tangent frame

bundle,  $E = Q \times_X Fr(X)$ , and  $\pi_i : X^l \rightarrow X$  projection onto the  $i$ -th factor.

$$Gl(\Sigma, \alpha) = \prod_{i=1}^l \pi_i^* E \times_{SO(4)} Z_{n_i}^s(\epsilon) / S^1$$

This is a bundle over  $X^l$ , to get a bundle over  $\Sigma$ , we have to mod out by  $P(n_1, \dots, n_l)$  the subgroup of the symmetric group on  $l$  elements preserving the multiplicities  $n_i$ . Our neighborhood is then

$$\nu_\alpha = (\mathbb{C}^N \times_{\Gamma_\alpha} Gl(\Sigma, \alpha)) / P(n_1, \dots, n_l).$$

$\Gamma_\alpha$  acts by diagonal multiplication on the factors of  $Q$  in each  $E$ . This space is fibered by  $\pi_\Sigma : \nu_\alpha \rightarrow \mathbb{C}^N / \Gamma_\alpha \times (\Sigma \setminus \nu(\Delta))$ .

In [T], Taubes describes a homeomorphism from these spaces of gluing data,  $\nu_\alpha$  onto their image in the moduli space. We divide this map into two parts, a splicing map and a perturbation of the image of the splicing map onto the moduli space. The splicing map  $\gamma'_\Sigma$  describes how the background connection in  $\mathbb{C}^N / \Gamma_\alpha$  and the instantons on  $S^4$  are spliced together using cut-off functions and the frames to give an almost anti-self dual connection. The perturbation, which we write as  $\xi$ , moves the image of  $\gamma'_\Sigma$  onto the parametrized moduli space. Note that the element of  $\mathbb{C}^N / \Gamma_\alpha$  describes a connection and a metric, so the gluing is performed with respect to the appropriate metric. The composition,  $\gamma_\Sigma = \gamma'_\Sigma + \xi_\gamma$ , is called the gluing map. The union of the images of  $\nu_\alpha$  cover a neighborhood of  $\mathbb{C}^N / \Gamma_\alpha \times \Sigma^r(X)$  and we can define a link of these reducibles in this neighborhood. This defines a homology class  $[D(\alpha, N, g)]$  ( $g$  being the metric with respect to which  $[\alpha]$  is anti-self dual). The difference term, for  $\alpha^2 < -1$  is calculated by

$$\delta_P(\alpha)(z) = \langle \bar{\mu}(z), [D(\alpha, N, g)] \rangle.$$

For  $\alpha^2 = -1$ , there is an obstruction to gluing. We write  $e(O)$  for the Euler class of this bundle and our calculation can be expressed as

$$\delta_P(\alpha)(z) = \langle \bar{\mu}(z), e(O) \cap [D(\alpha, N, g)] \rangle.$$

On the complement of the reducibles in the image of  $\nu_\alpha$ , the  $\Gamma_\alpha$  action on  $\mathbb{C}^N \times Gl(\Sigma, \alpha)$  is free. We write  $c_1$  for the first Chern class of the  $\Gamma_\alpha$  action. For a homology class  $x$  on  $X$ , we write  $\Sigma(x)$  to denote the cohomology class on  $\Sigma$  that, under the map  $X^n \rightarrow \Sigma$ , pulls back to the symmetrization of  $\pi_i^* PD[x]$ . From [KoM], we have the following description of the  $\mu$  map on the images of  $\gamma_\Sigma$ .

**Lemma 1.1.** *For  $x_i \in H_2(X; \mathbb{Z})$  and  $1 \in H_0(X; \mathbb{Z})$  the generator,*

$$\begin{aligned} \gamma_\Sigma^* \bar{\mu}(x) &= \frac{1}{2} \langle \alpha, x \rangle c_1 + \pi_\Sigma^* \Sigma(x) \\ \gamma_\Sigma^* \bar{\mu}(1) &= -\frac{1}{4} c_1^2 + \pi_\Sigma^* \Sigma(1) \\ \gamma_\Sigma^* (e(O)) &= c_1 \end{aligned}$$

The difficulties in computing these pairings arise from two sources. The first is the presence of  $Z_k^s(\epsilon)$  which remain mysterious in spite of the ADHM description. The second is the problem of the overlap of the sets  $\partial \nu_\alpha$ . For  $r = 0, 1$ , neither of these difficulties arise. The main result here is a method for dealing with these problems in the next simplest situation, where two strata are present.

## 2: First Order Reducibles

We begin our work by calculating the case where the reducible has only one point of totally concentrated curvature. This computation has already been performed in the  $SU(2)$  case in [Y]. We note that  $Z_1^S(\epsilon) \simeq c(SO(3))$ . The  $SO(4) = SU(2)_L \times_{\mathbb{Z}/2} SU(2)_R$  rotation group acts on this via its projection to  $SO(3)_R$  (see [Y]). We may thus simplify  $Fr(X) \times_{SO(4)} c(SO(3)) \simeq Fr \wedge_+^2 TX \times_{SO(3)} c(SO(3))$  and describe  $\nu_\alpha$  as:

$$\nu_\alpha \simeq \Gamma_\alpha \setminus (\mathbb{C}^N \times Q_\alpha \times_X Fr \wedge_+^2 TX \times c(SO(3))/S^1)$$

We present a covering of this space by a vector bundle. Let  $Q_{\frac{\alpha}{2}} \rightarrow Q_\alpha$  be a twofold fiberwise covering and  $\tilde{\wedge}_+^2 \rightarrow Fr \wedge_+^2 TX$  be an  $SU(2)$  lifting. Although these bundles need not exist on  $X$ , one can pull back to  $X' = X \times_{K(\mathbb{Z}, 2)} K(\mathbb{Z}, 2)$  where  $X \rightarrow K(\mathbb{Z}, 2)$  is given by  $\alpha$  and  $K(\mathbb{Z}, 2) \rightarrow K(\mathbb{Z}, 2)$  is given by doubling the universal class. Since  $\alpha$  is an integral lift of  $w_2(P)$ , pulling back to  $X'$  kills  $w_2$ . Moreover, the map  $X' \rightarrow X$  is a rational homotopy isomorphism, we can either pull all the bundles back to  $X'$ , work there and divide by the degree, or simply work on  $X$  pretending that these liftings exist. We thus cover  $\nu_\alpha$  with

$$S^1 \setminus (\mathbb{C}^N \times Q_{\frac{\alpha}{2}} \times_X \tilde{\wedge}_+^2 \times_{SU(2)} \mathbb{C}^2/S^1).$$

The degree of this covering map is  $-2^N$ : the map  $\mathbb{C}^2 \rightarrow c(SO(3))$  has degree  $-2$  (see [Oz]),  $Q_{\frac{\alpha}{2}} \rightarrow Q_\alpha$  has degree 2 and to make the covering equivariant with respect to the stabilizer action, we have to square each of the  $\mathbb{C}$ 's in  $\mathbb{C}^N$ . The stabilizer action doubles, subtracting one factor of two from the degree and pulling back  $c_1$  of the  $\Gamma_\alpha$  action to  $2h$  where  $h$  is the  $c_1$  of the lefthand  $S^1$  action upstairs. We note that the cover of  $\partial\nu_\alpha$  can be written as  $\mathbb{P}(\mathbb{C}^N \oplus \tilde{E})$  where  $\tilde{E} \simeq Q_{\frac{\alpha}{2}} \otimes (\tilde{\wedge}_+^2 \times_{SU(2)} \mathbb{C}^2)$ . If  $p_+ = p_1(Fr \wedge_+^2 TX)$ , then  $c_1(\tilde{E}) = \alpha$  and  $c_2(\tilde{E}) = \frac{1}{4}(\alpha^2 - p_+)$ . If we write  $b_i = \langle \alpha, x_i \rangle$ , our computation is now

(2.1)

$$\begin{aligned} \delta_P(\alpha)(x_1, \dots, x_{d-2l}, 1^l) &= \frac{(-1)}{2^N} \left\langle \left( \prod_{i=1}^{d-2l} (b_i h + \pi^* PD[x_i]) \right) (-h^2 + \pi^* PD[1])^l, [\mathbb{P}(\mathbb{C}^N \oplus \tilde{E})] \right\rangle \\ &= \frac{(-1)^{l+1}}{2^N} \left\langle \left( \prod_{j=1}^{d-2l} b_j \right) h^d + \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) \pi^* PD[x_i] h^{d-1} + \frac{1}{2} \sum_{r,s=1}^{d-2l} \left( \prod_{i \neq r,s} b_i \right) \pi^* PD[x_r] \pi^* PD[x_s] h^{d-2} \right. \\ &\quad \left. - l \left( \prod_{j=1}^{d-2l} b_j \right) h^{d-2} \pi^* PD[1], [\mathbb{P}(\mathbb{C}^N \oplus \tilde{E})] \right\rangle \end{aligned}$$

We use the definition of characteristic class on this projectivization:  $h^{N+2} - \pi^* c_1(\tilde{E}) h^{N+1} + \pi^* c_2(\tilde{E}) h^N = 0$  and  $d = N + 3$  to rewrite the above equation as

$$\begin{aligned} (2.2) \quad &= \frac{(-1)^{l+1}}{2^N} \left\langle \left( \prod_{j=1}^{d-2l} b_j \right) \left( \pi^* \alpha^2 - \frac{1}{4}(\alpha^2 - p_+) h^{N+1} + \sum_{i=1}^{d-2l-1} \left( \prod_{j \neq i} b_j \right) \pi^* \alpha \pi^* PD[x_i] h^{N+1} \right. \right. \\ &\quad \left. \left. + \frac{1}{2} \sum_{r,s=1}^{d-2l} \left( \prod_{i \neq r,s} b_i \right) h^{N+1} \pi^* PD[x_r] \pi^* PD[x_s] - l \left( \prod_{j=1}^{d-2l} b_j \right) h^{N+1} \pi^* PD[1], [\mathbb{P}(\mathbb{C}^N \oplus \tilde{E})] \right\rangle \right. \end{aligned}$$

Each of these terms contains a factor pulled up from  $X$  dual to a point. Thus, we have reduced our computation to the pairing  $\langle h^{N+1}, \mathbb{C}P^{N+1} \rangle = (-1)^{N+1}$ . We further simplify by noting that  $\alpha^2 = p_1 + 4 = 1 - d$ . The calculation for the case  $\alpha^2 = -1$  is done similarly. Our answer is

**Proposition 2.1.** *For a reducible connection  $\alpha$  with  $\alpha^2 = p_1 + 4$ ,  $\alpha^2 < -1$*

$$\begin{aligned} \delta_P(\alpha)(x_1, \dots, x_{d-2l}, 1^l) \\ = \left( \left( -\frac{1}{2} \right)^{d+l-1} ((d+p_+ + 3 - 12l)\alpha^{d-2l}) + \frac{(-1)^{d-1}(d-2l)!}{2^{d-3}2!(d-2l-2)!} q\alpha^{d-2l-2} \right) (x_1, \dots, x_{d-2l}) \\ = \frac{(-1)^{d+l-1}}{2^d} ((2d+2p_+ + 6 - 24l)\alpha^{d-2l} + 4(d-2l)(d-2l-1)q\alpha^{d-2l-2}) (x_1, \dots, x_{d-2l}) \end{aligned}$$

If  $\alpha^2 = -1$ , so  $d = 2$  we have

$$\begin{aligned} \delta_P(\alpha)(x_1, x_2) &= \left( \frac{1}{4} \right) ((5+p_+)\alpha^2 + 4q) (x_1, x_2) \\ \delta_P(\alpha)(1) &= -\frac{1}{4}(2p_+ - 14) \end{aligned}$$

**Remark:** This calculation agrees with that of Friedman and Qin in [FQ, Thm 6.4, 6.5] for the cases  $l = 0$  and  $l = 1$ . The co-efficient of  $q\alpha^{d-2l-2}$  agrees with that of [KoM] up to a sign error in [KoM] as noted in [KoL].

### 3: Second Order Reducibles

We now consider the case where  $\alpha^2 - p_1 = 8$ , where the set of reducibles is described by  $[\alpha] \times \Sigma^2(X)$ . This set is covered by two strata, the off-diagonal  $\Sigma_{(1,1)}$  and the diagonal  $\Sigma_2$ . The off-diagonal, or upper stratum, has a neighborhood described by two copies of the bundle described in section 2. We compactify this set by extending the bundles over the diagonal. The lower stratum involves the space of charge two instantons on  $S^4$ ,  $Z_2^s(\epsilon)$ . Applying an extension of some work of Hattori's, [Ha,Ha'], gives enough information about the equivariant cohomology of this space to compute the cup-products on this space. We give a natural compactification of this set and finally show the error term, resulting from the two compactifications, vanishes. Our result is:

**Proposition 3.0.** *For  $\alpha^2 = p_1 + 8$ , and  $\alpha^2 < -1$ ,*

$$\begin{aligned} (-1)^{d+l} 2^d \delta_P(\alpha)(x_1, \dots, x_{d-2l}, 1^l) \\ = (2d^2 + 4dp_+ + 2p_+^2 + 13d + 10p_+ + 21 + l(-408 - 48d - 48p_+ + 288l)) \alpha^{d-2l} (x_1, \dots, x_{d-2l}) \\ + (16p_+ + 16d + 32 - 192l) (\alpha^{d-2l-2} q) (x_1, \dots, x_{d-2l}) \\ + 8 \frac{(d-2l)!}{(d-2l-4)!} \alpha^{d-2l-4} q^2 \end{aligned}$$

If  $\alpha^2 = -1$ , so that  $d = 6$  and  $0 \leq l \leq 3$ ,

$$\begin{aligned} (-1)^{l+3} 2^6 \delta_P(\alpha)(x_1, \dots, x_{6-2l}, 1^l) \\ = (2p_+^2 - 58p_+ + 171 - 344l + 16lp_+ + 160l^2) \alpha^{6-2l} \\ + (16p_+ + 128 - 192l) \alpha^{4-2l} q + 8 \frac{(6-2l)!}{(2-2l)!} \alpha^{2-2l} q^2 \end{aligned}$$

where the terms with negative exponents will vanish.

**Remark:** This computation agrees with those of [EG] and [FQ].

### 3.1: The Higher Stratum for $r=2$

We pull the bundle  $Gl(\Sigma_{(1,1)}, \alpha)$  back to  $X^2$  and extend it over the diagonal, writing this compactification as  $\nu_{\Sigma_{(1,1)}} \cup C$ . Both factors of  $E$  are covered with  $\tilde{E}$ . This branched cover has degree  $2^N$  from the factors of  $\mathbb{C}$ ,  $2^2$  from the two copies of  $E$ , 2 from  $X^2 \rightarrow \Sigma^2 X$  and  $\frac{1}{2}$  from the doubling of the  $\Gamma_\alpha$  action. Thus, at a price of a factor of  $2^{-N-2}$  we can pair our cohomology classes with  $\mathbb{P}(\mathbb{E}) = \mathbb{P}(\mathbb{C}^N \oplus \pi_1^* \tilde{E} \oplus \pi_2^* \tilde{E})$  as in the previous section. In the following computation, we relax our notation, writing  $x_i$  and 1 for both the homology classes and their Poincare duals.

(3.1.1)

$$\begin{aligned}
& < \prod_{i=1}^{d-2l} \overline{\mu}(x_i) \cup \overline{\mu}(1)^l, [\nu_{\Sigma_{(1,1)}} \cup C] > = 2^{-N-2} < \left( \prod_{i=1}^{d-2l} (b_i h + \pi_1^* x_i + \pi_2^* x_i) \right) (-h^2 + \pi_1^* 1 + \pi_2^* 1)^l, [\mathbb{P}(\mathbb{E})] > \\
& = \frac{(-1)^l}{2^{N+2}} < \left( \prod_{j=1}^{d-2l} b_j \right) h^d + h^{d-1} \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (\pi_1^* x_i + \pi_2^* x_i) \\
& \quad + h^{d-2} \left[ \left( \prod_{j=1}^{d-2l} b_j \right) (-l)(\pi_1^* 1 + \pi_2^* 1) + \sum_{|I|=2}^d \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right] \\
& \quad + h^{d-3} \left[ \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (\pi_1^* x_i + \pi_2^* x_i) (-l)(\pi_1^* 1 + \pi_2^* 1) + \sum_{|I|=3}^d \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right] \\
& \quad + h^{d-4} \left[ \left( \prod_{j=1}^{d-2l} b_j \right) l(l-1) \pi_1^* 1 \pi_2^* 1 + \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) (-l)(\pi_1^* 1 + \pi_2^* 1) \right] \\
& \quad + h^{d-4} \left[ \sum_{|I|=4} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right], [\mathbb{P}(\mathbb{E})] >
\end{aligned}$$

Here these sums are over all  $I \subset \{1, \dots, d-2l\}$  of the appropriate size. Now,

(3.1.2)

$$\begin{aligned}
c_1(\mathbb{E}) &= \pi_1^* \alpha + \pi_2^* \alpha \\
c_2(\mathbb{E}) &= \pi_1^* \alpha \pi_2^* \alpha + \frac{1}{4} (\pi_1^* (\alpha^2 - p_+) + \pi_2^* (\alpha^2 - p_+)) \\
c_3(\mathbb{E}) &= \pi_1^* \alpha \left( \frac{1}{4} \pi_2^* (\alpha^2 - p_+) \right) + \pi_2^* \alpha \left( \frac{1}{4} \pi_1^* (\alpha^2 - p_+) \right) \\
c_4(\mathbb{E}) &= \frac{1}{16} \pi_1^* (\alpha^2 - p_+) \pi_2^* (\alpha^2 - p_+)
\end{aligned}$$

The Leray-Hirsch relationship is, using  $p_1 = -d - 3, \alpha^2 = p_1 + 8, \alpha^2 = -N - 2$  to get  $N = d - 7$

$$(3.1.3) \quad h^{d-3} - \pi^* c_1(\mathbb{E}) h^{d-4} + \pi^* c_2(\mathbb{E}) h^{d-5} - \pi^* c_3(\mathbb{E}) h^{d-6} + \pi^* c_4(\mathbb{E}) h^{d-7} = 0$$

This allows us to reduce the terms with  $h^k$  in them as in the  $r = 1$  case, writing  $c_i$  for  $\pi^* c_i(\mathbb{C}^N \oplus \tilde{E} \oplus \tilde{E})$ :

$$\begin{aligned}
 (3.1.4) \quad &= \frac{(-1)^l}{2^{N+2}} < \left( \prod_{j=1}^{d-2l} b_j \right) (c_1^4 - 3c_1^2 c_2 + 2c_1 c_3 + c_2^2 - c_4) h^{d-4} + (c_1^3 - 2c_1 c_2 + c_3) h^{d-4} \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (\pi_1^* x_i + \pi_2^* x_i) \\
 &+ (c_1^2 - c_2) h^{d-4} \left[ \left( \prod_{j=1}^{d-2l} b_j \right) (-l)(\pi_1^* 1 + \pi_2^* 1) + \sum_{|I|=2}^d \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right] \\
 &+ c_1 h^{d-4} \left[ \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (\pi_1^* x_i + \pi_2^* x_i) (-l)(\pi_1^* 1 + \pi_2^* 1) + \sum_{|I|=3} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right] \\
 &+ h^{d-4} \left[ \left( \prod_{j=1}^{d-2l} b_j \right) l(l-1) \pi_1^* 1 \pi_2^* 1 + \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) (-l)(\pi_1^* 1 + \pi_2^* 1) \right] \\
 &+ h^{d-4} \left[ \sum_{|I|=4} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right], [\mathbb{P}(\mathbb{E})] >
 \end{aligned}$$

In the above equation, each term contains an eight dimensional cohomology class pulled up from  $X^2$ . Cupping with these terms is equivalent to restricting to a fiber of  $\mathbb{P}(E)$  which is  $\mathbb{C}P^{N+4} = \mathbb{C}P^{d-4}$ . The term  $h^{d-4}$  gives us a factor of  $(-1)^{d-4}$ , and all that remains is to calculate the intersection numbers of the classes pulled up from  $X^2$ . We use the notation  $p_+ = p_1(Fr \wedge_+^2 TX)$ .

$$\begin{aligned}
 (3.1.5) \quad &< \prod_{i=1}^{d-2l} \overline{\mu}(x_i) \cup \overline{\mu}(1)^l, [\nu_{\Sigma_{(1,1)}} \cup C] > \\
 &= \frac{(-1)^{d+l-4}}{2^{d-5}} < \left( \prod_{j=1}^{d-2l} b_j \right) \left( \frac{9}{16} \pi_1^* \alpha^2 \pi_2^* \alpha^2 + \frac{3}{16} (\pi_1^* \alpha^2 \pi_2^* p_+ + \pi_1^* p_+ \pi_2^* \alpha^2) + \frac{1}{16} \pi_1^* p_+ \pi_2^* p_+ \right) \\
 &+ \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) \left( \frac{3}{4} (\pi_1^* \alpha \pi_2^* \alpha^2 + \pi_1^* \alpha^2 \pi_2^* \alpha) + \frac{1}{4} (\pi_1^* \alpha \pi_2^* p_+ + \pi_1^* p_+ \pi_2^* \alpha^2) \right) (\pi_1^* x_i + \pi_2^* x_i) \\
 &+ \left[ \left( \prod_{j=1}^{d-2l} b_j \right) (-l)(\pi_1^* 1 + \pi_2^* 1) + \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right] \left[ \frac{3}{4} (\pi_1^* \alpha^2 + \pi_2^* \alpha^2) + \pi_1^* \alpha \pi_2^* \alpha + \frac{1}{4} (\pi_1^* p_+ + \pi_2^* p_+) \right] \\
 &+ \left[ \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (\pi_1^* x_i + \pi_2^* x_i) (-l)(\pi_1^* 1 + \pi_2^* 1) + \sum_{|I|=3} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right] (\pi_1^* \alpha + \pi_2^* \alpha) \\
 &+ \left[ \left( \prod_{j=1}^{d-2l} b_j \right) l(l-1) \pi_1^* 1 \pi_2^* 1 + \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) (-l)(\pi_1^* 1 + \pi_2^* 1) \right] \\
 &+ \left[ \sum_{|I|=4} \left( \prod_{j \notin I} b_j \right) \prod_{i \in I} (\pi_1^* x_i + \pi_2^* x_i) \right], [X^2] >
 \end{aligned}$$

Now,  $< \alpha^2, [X] > = p_1 + 8 = 5 - d$ , and using the following combinatorial lemma we can calculate the intersections on  $X^2$ . ■

**Lemma 3.1.1.** *On  $X^r$ , for  $|I| = 2r$ ,  $x_i, \alpha \in H_2(X)$ ,*

$$\prod_{i \in I} \left( \sum_{j=1}^r (\pi_j^* x_i) \right) = \frac{(2r)!}{2^r} q^{(r)}(x_{i_1}, \dots, x_{i_{2r}}) [X^r]$$

$$q^r(x_1, \dots, x_{2r-1}, \alpha_*) = (q^{r-1} \alpha)(x_1, \dots, x_{2r-1})$$

When the dust settles, our equation becomes.

$$(3.1.6) \quad \begin{aligned} & < \prod_{i=1}^{d-2l} \overline{\mu}(x_i) \cup \overline{\mu}(1)^l, [\nu_{\Sigma_{(1,1)}} \cup C] > \\ &= \frac{(-1)^{d+l-4}}{2^{d-5}} \left( \prod_{j=1}^{d-2l} b_j \right) \left[ \frac{9}{16}(5-d)^2 + \frac{3}{8}(5-d)p_+ + \frac{1}{16}p_+^2 + \frac{3}{2}(d-2l)(5-d) \right. \\ &\quad \left. + \frac{1}{2}(d-2l)p_+ - \frac{3}{2}l(5-d) - \frac{1}{2}lp_+ + 2 \binom{d-2l}{2} - 2l(d-2l) + l(l-1) \right] \\ &\quad + \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) \left[ \frac{3}{2}(5-d)q(x_{i_1}, x_{i_2}) + \frac{1}{2}q(x_{i_1}, x_{i_2})p_+ - 2lq(x_{i_1}, x_{i_2}) \right] \\ &\quad + \sum_{|I|=3} \left( \prod_{j \notin I} b_j \right) \frac{4!}{4} q^2(x_{i_1}, x_{i_2}, x_{i_3}, \alpha) + \sum_{|I|=4} \left( \prod_{j \notin I} b_j \right) \frac{4!}{4} q^2(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) \end{aligned}$$

After this, some work with symmetric polynomials gives us the answer. Similarly brutal work gives us a result for reducibles with  $\alpha^2 = -1$ .

**Proposition 3.1.2.** *For a reducible connection with  $p_1 + 8 = \alpha^2 < -1$*

$$\begin{aligned} (-1)^{d+l} 2^d &< \prod_{i=1}^{d-2l} \overline{\mu}(x_i) \cup \overline{\mu}(1)^l, [\nu_{\Sigma_{(1,1)}} \cup C] > \\ &= [450 + 28d + 60p_+ + 2d^2 + 4dp_+ + 2p_+^2 + l(-688 - 48d + -48p_+ + 288l)] \alpha^{d-2l} (x_1, \dots, x_{d-2l}) \\ &\quad + [16d + 16p_+ + 112 - 192l] \binom{d-2l}{2} \alpha^{d-2} q(x_1, \dots, x_d) \\ &\quad + 8 \frac{(d-2l)!}{(d-2l-4)!} \alpha^{d-2l-4} q^2(x_1, \dots, x_{d-2l}) \end{aligned}$$

For a reducible with  $\alpha^2 = -1$ ,  $d = 6$

$$\begin{aligned} & (-1)^{l+3} 2^6 \delta_P(\alpha)(x_1, \dots, x_{d-2l}, 1^l) \\ &= [690 - 108p_+ + 2p_+^2 - 624l + 16lp_+ + 160l^2] \alpha^{6-2l} \\ &\quad + [16p_+ + 208 - 192l] \binom{6-2l}{2} q\alpha^{4-2l} + \frac{(6-2l)!}{(2-2l)!} 8q^2 \alpha^{2-2l} \end{aligned}$$

where the terms  $q\alpha^{4-2l}$  and  $q^2\alpha^{2-2l}$  vanish if  $l > 2$  and  $l > 1$ , respectively.

### 3.2: The Lower Stratum

The lower stratum of  $[\alpha] \times \Sigma^2 X$  has a neighborhood described by

$$\nu_2 = \Gamma_\alpha \setminus (\mathbb{C}^N \times (Q_\alpha \times_X \text{Fr}(X) \times_{SO(4)} Z_2^s(\epsilon)/S^1)).$$

On this set,  $\overline{\mu}(x_i) = \frac{b_i}{2}c_1 + 2\pi^*x_i$ , where  $c_1$  is the first Chern class of the  $\Gamma_\alpha$  action,  $b_i = \langle \alpha, x_i \rangle$ . Note that the factor of two results from calculating the restriction to the diagonal of  $1 \times \pi_2^*x_i + \pi_1^*x_i \times 1$ . Similarly  $\overline{\mu}(1) = -\frac{1}{4}c_1^2 + 2\pi^*1$ . Let  $T \subset Z_2(\epsilon)$  be the set of generalized connections where the background connection is trivial. We also have  $T \subset Z_2^s(\epsilon)$  as the  $SO(3)$  action on the frame is trivial on this set. The set of reducibles in  $\nu_2$  is then parametrized by

$$\nu_2(T) = \Gamma_\alpha \setminus (Q_\alpha \times_X \text{Fr}(X) \times_{SO(4)} T) / S^1 = \text{Fr}(X) \times_{SO(4)} T.$$

The set  $T = D^4/(\mathbb{Z}/(2))$  is not compact, so the link of  $\nu_2(T)$  in  $\nu_2$  will not be compact. This non-compactness arises from two singular points moving away from the diagonal: i.e. into  $\nu_{(1,1)}$  the gluing data associated to the upper stratum. The link of  $\nu_2(T)$  can be presented as the union of two sets. Let  $L^s \subset Z_2^s(\epsilon)$ ,  $L \subset Z_2(\epsilon)$  be the links of  $T$  in these sets. Our two open sets are then,

$$\begin{aligned} U_1 &= \Gamma_\alpha \setminus (S^{2N-1} \times (Q_\alpha \times_X \text{Fr}(X) \times_{SO(4)} Z_2^s(\epsilon)/S^1)), \\ U_2 &= \Gamma_\alpha \setminus (\mathbb{C}^N \times (Q_\alpha \times_X \text{Fr}(X) \times_{SO(4)} L^s/S^1)). \end{aligned}$$

The set  $U_1$  is where the connection on  $S^4$  is allowed to approach the cone point, but the background connection is kept away from  $T$ , while the set  $U_2$  is where the background connection can approach the reducible, but the  $S^4$  connection may not approach  $T$ . The boundary of  $T$  is then covered by  $U_2$ . We write  $\partial L^s$  for the boundary of  $L^s$ . Now, the  $SO(4)$  action on  $Z_2^s(\epsilon)$  respects strata and commutes with the framing action. Thus, we can compactify  $L^s$  by taking cones on the  $SU(2)_L$  orbits in  $\partial L^s$ . Write  $L^s \cup C_2^s$ , for this compactification. The framing action extends over this compactification, so  $L^s \cup C_2^s$  is an  $SO(3)$  bundle over  $L \cup C_2$  the space constructed by taking cones on the  $SU(2)_L$  orbits in  $\partial L$ . We then use this compactification to compactify  $U_2$ , replacing  $L^s$  with  $L^s \cup \tilde{C}_s$ . Write  $\overline{U}_2$  for this compactification. We see that the cohomology classes will extend over this compactification as they can be expressed as characteristic classes of the equivariant frame bundle  $L^s \cup \tilde{C}_2$ .

The pairing of the cohomology classes with  $U_1 \cup \overline{U}_2$  can be written as

$$\begin{aligned} (3.2.1) \quad & < \prod_{i=1}^{d-2l} \left( \frac{b_i}{2}c_1 + 2\pi^*x_i \right) \left( -\frac{1}{4}c_1^2 + 2\pi^*1 \right)^l, [U_1 \cup \overline{U}_2] > \\ &= 2^{-d+2l} \left[ \left( \prod_{j=1}^{d-2l} b_j \right) c_1^{d-2l} + 4 \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) c_1^{d-2l-1} \pi^* x_i \right. \\ &\quad \left. + 16 \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) c_1^{d-2l-2} q(x_{i_1}, x_{i_2}) \right] (-4)^{-l} (c_1^2 - 8\pi^*1)^l, [U_1 \cup \overline{U}_2] > \\ &= \frac{(-1)^l}{2^d} < \left( \prod_{j=1}^{d-2l} b_j \right) c_1^d + 4 \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) c_1^{d-1} x_i \\ &\quad + \left( 16 \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) q(x_{i_1}, x_{i_2}) - 8l \left( \prod_{j=1}^{d-2l} b_j \right) \right) \pi^* 1 c_1^{d-2}, [U_1 \cup \overline{U}_2] > \end{aligned}$$

We now give an argument that we can eliminate the set  $U_1$  from our consideration. We note that the set  $\overline{U}_2$  can be seen as a vector bundle over a space  $B$ :

$$\begin{array}{c} \mathbb{C}^N \times_{\Gamma_\alpha} (Q_\alpha \times_X Fr(X) \times_{SO(4)} (L^s \cup C_2^s) / S^1) \\ \pi \downarrow \\ B = \Gamma_\alpha \setminus (Q_\alpha \times_X Fr(X) \times_{SO(4)} (L^s \cup C_2^s) / S^1) \end{array}$$

The Thom class of this bundle is  $(-c_1)^N$ , the negative sign appearing because Since  $Z_2^s(\epsilon)$  is equivariantly (with respect to the framing and rotation actions) retractable onto a point in  $T$ ,  $U_1$  retracts onto  $\mathbb{C}P^{N-1} \times X$ . Here  $c_1$  is the negative of the hyperplane section. Thus  $c_1^N$  vanishes on this open set. This tells us  $(-c_1)^N$  is the Thom class of the space  $B$ . Formally, we have

**Lemma 3.2.1.** *For a cohomology class  $x$  of appropriate dimension, and  $i : B \rightarrow U_1 \cup \overline{U}_2$ ,*

$$\langle x \cup (-c_1)^N, [U_1 \cup \overline{U}_2] \rangle = \langle i^* x, [B] \rangle.$$

Because  $d - 2 > N - 1$ , we see each term in equation 3.2.1 is divisible by the Thom class  $(-c_1)^N$  and we can use Lemma 3.2.1. Now, the base space  $B$  can be rewritten as

$$\Gamma_\alpha \setminus (Q_\alpha \times_X Fr(X) \times_{SO(4)} (L^s \cup C_2^s) / S^1) \simeq Fr(X) \times_{SO(4)} (L^s \cup C_2^s) / S^1$$

Prior to taking the  $\Gamma_\alpha$  action, the above could be identified with the tensor product of  $\pi^* Q_\alpha$  with the inverse of the  $S^1$  bundle  $Fr(X) \times_{SO(4)} (L^s \cup C_2^s) \rightarrow Fr(X) \times_{SO(4)} (L^s \cup C_2^s) / S^1$ . We write  $h$  for the first chern class of this latter bundle. The inverse arises because  $\Gamma_\alpha$  acts on the  $Q_\alpha$  fibers positively and on this  $S^1$  bundle by inverses via the  $S^1$  quotient. The  $c_1$  of the  $\Gamma_\alpha$  action then becomes  $\pi^* \alpha - h$ . Our pairing has become

(3.2.2)

$$\begin{aligned} &= \frac{(-1)^{l+N}}{2^d} \left\langle \left( \prod_{j=1}^{d-2l} b_j \right) (\pi^* \alpha - h)^{d-N} + 4 \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (\pi^* \alpha - h)^{d-N-1} x_i \right. \\ &\quad \left. + \left( 16 \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) q(x_{i_1}, x_{i_2}) - 8l \left( \prod_{j=1}^{d-2l} b_j \right) \pi^* 1 \right) (\pi^* \alpha - h)^{d-N-2}, [Fr(X) \times_{SO(4)} (L^s \cup C_2^s) / S^1] \right\rangle \\ &= \frac{(-1)^{l+d}}{2^d} \left\langle \left( \prod_{j=1}^{d-2l} b_j \right) (h^7 - 7h^6 \pi^* \alpha + 21h^5 \pi^* \alpha^2) - 4 \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (h^6 - 6h^5 \pi^* \alpha) \pi^* x_i \right. \\ &\quad \left. + \left( 16 \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) q(x_{i_1}, x_{i_2}) - 8l \left( \prod_{j=1}^{d-2l} b_j \right) \right) \pi^* 1 h^5, [Fr(X) \times_{SO(4)} (L^s \cup C_2^s) / S^1] \right\rangle \end{aligned}$$

We can push these classes down from the  $S^1$  quotient to the  $SO(3)$  quotient of  $Fr(X) \times_{SO(4)} (L^s \cup \tilde{C}_2) \setminus T$ . This turns  $h^r$  into  $2\wp^{\frac{r-1}{2}}$  if  $r$  is odd and makes  $h^r$  vanish if  $r$  is even where  $\wp$  is the  $p_1$  of the  $SO(3)$  action. Our pairing is now:

$$\begin{aligned} (3.2.3) \quad &= \frac{(-1)^{l+d}}{2^d} \left\langle \left( \prod_{j=1}^{d-2l} b_j \right) (2\wp^3 + 42\wp^2 \pi^* \alpha^2) - 4 \sum_{i=1}^{d-2l} \left( \prod_{j \neq i} b_j \right) (-12\wp^2 \pi^* \alpha) \pi^* x_i \right. \\ &\quad \left. + \left( 16 \sum_{|I|=2} \left( \prod_{j \notin I} b_j \right) q(x_{i_1}, x_{i_2}) - 8l \left( \prod_{j=1}^{d-2l} b_j \right) \right) 2\pi^* 1 \wp^2, [Fr(X) \times_{SO(4)} (L^s \cup C_2)] \right\rangle \end{aligned}$$

We prove the following proposition in the next section

**Proposition 3.3.19.** *For  $p : ESO(4) \times_{SO(4)} (L \cup C_2) \rightarrow BSO(4)$  and  $p_+ = p_1 + 2e$ ,*

$$\begin{aligned} p_*(\wp^2) &= -\frac{5}{2} \\ p_*(\wp^3) &= -p_- 20p_+ \end{aligned}$$

Pulled back to a manifold with  $b^+ = 1$ ,  $p_*(\wp^3) = 48 - 25p_+$ .

Plugging this in we have

**Proposition 3.2.2.** *If  $\alpha^2 < -1$ , and for the choice of compactly supported lift of the cohomology class given by adding on  $C_2$ ,*

$$\begin{aligned} (-1)^{d+l} 2^d &< \prod_{i=1}^{d-2l} \overline{\mu}(x_i) \cup \overline{\mu}(1)^l, [\partial\nu_{\Sigma_2}] > \\ &= \left( (-32p_+ - 15d - 429 + 280l)\alpha^{d-2l} - 80 \binom{d-2l}{2} \alpha^{d-2} q \right) (x_1, \dots, x_d) \end{aligned}$$

If  $\alpha^2 = -1$ , then

$$(-1)^{l+7} 2^6 \delta_P(\alpha)(x_1, \dots, x_{6-2l}, 1^l) = (-50p_+ - 519 + 280l)\alpha^{6-2l} - 80 \binom{6-2l}{2} \alpha^{4-2l} q$$

### 3.3: ADHM Calculations

In this section we use the ADHM description of  $Z_2^s(\epsilon)$  to prove Proposition 3.3.19. As this section is somewhat involved, we begin with a brief summary.

We decompose the link  $L$ , of the trivial strata  $T$  in  $Z_s(\epsilon)$  into two sets,  $A, B$ . The set  $A$  is branch covered by  $S^3 \times \mathbb{R}^+ \times S^4$ , which matches the gluing data bundle associated to  $\Sigma_{(1,1)}$  near the diagonal. The set  $B \simeq (S^3 \times S^2) * S^1 / S^1$  shows why there is more to  $Z_2(\epsilon)$  than just gluing in two charge one instantons. In Lemma 3.3.4, the link is then described as a mapping cone  $\partial A \rightarrow B$ , with the factor of  $\mathbb{R}^+$  in  $A$  serving as the cone factor. Because  $\partial L$  is contained in  $A$ , we are able to describe the cap given by taking cones on  $SO(3)_L$  orbits from our knowledge of the  $SO(4)$  action on  $A$ , the cap turns out to be branch covered by  $D^4 \times S^4$ . In Lemmas 3.3.5-9, we assemble information about the cohomology of these spaces. In Lemmas 3.3.11-15, we calculate the restriction of  $\wp$  to  $L$  and to the cap. Lemma 3.3.16 allows us to take unique lifts of  $\wp^2$  to cohomology classes with compact support on  $L$  and on the cap. The remainder of the section contains the calculations using this.

We begin by recalling the ADHM description of the moduli space for charge  $k = 2$  (see [A,BoM,Ha]).

$$Z_2^s(\epsilon) = \left\{ \begin{pmatrix} \lambda_1 & \lambda_2 \\ a & c \\ c & -a \end{pmatrix} : \lambda_i, a, c \in \mathbb{H}, \text{Im}(\overline{\lambda}_1 \lambda_2) = 2\text{Im}(\overline{c}a) \right\} / O(2)$$

Here,  $O(2)$  acts by

$$\begin{pmatrix} \Lambda \\ B \end{pmatrix} \rightarrow \begin{pmatrix} \Lambda T \\ T^t B T \end{pmatrix}$$

We work with the double branched cover given by only modding out by  $SO(2)$  at the expense of a factor of 2. We refer to this space as  $\tilde{Z}_2^s(\epsilon)$ . Then the  $SO(2)$  action is given by

$$(\lambda_1 \quad \lambda_2) \rightarrow (\lambda_1 \quad \lambda_2) \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} = (\lambda_1 \quad \lambda_2) R(\theta)$$

and

$$(a \quad c) \rightarrow (a \quad c) \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ \sin(2\theta) & \cos(2\theta) \end{pmatrix} = (a \quad c) R(2\theta)$$

We note that the ADHM description is usually accompanied by a rank condition. Relaxing this rank condition, as done here, is equivalent to compactifying the moduli space (see [BoM, Thm. 4.4.3] or [DK, Cor 3.4.10]). Requiring  $\text{tr}(B) = 0$  is equivalent to the condition  $\int x_i |F_A|^2 dx = 0$  (see [Ma]). The ADHM correspondence is equivariant with respect to the  $SO(3)$  action on the frame and the  $SO(4)$  rotation action when the group actions are given on the ADHM data by (for  $q \in SO(3)$ ,  $(p_L, p_R) \in SU(2) \times_{\mathbb{Z}/2} SU(2)_R \simeq SO(4)$ )

$$\begin{pmatrix} \lambda_1 & \lambda_2 \\ a & c \\ c & -a \end{pmatrix} \rightarrow \begin{pmatrix} q\lambda_1 \bar{p}_R & q\lambda_2 \bar{p}_R \\ p_L a \bar{p}_R & p_L c \bar{p}_R \\ p_L c \bar{p}_R & -p_L a \bar{p}_R \end{pmatrix}$$

To keep our orientation, the standard  $SO(4)$  action on the northern hemisphere of  $S^4$ ,  $\mathbb{H}$ , is given by  $h \rightarrow p_L h \bar{p}_R$ . We will write  $\tilde{G}$  for  $SU(2)_L \times SU(2)_R$ . Note that even on  $\tilde{Z}_2^s(\epsilon)$ , the  $SO(2)$  action gives an equivalence between  $((a, c), (\lambda_1, \lambda_2))$  and  $((a, c), (-\lambda_1, -\lambda_2))$ . Thus the framing action on  $\tilde{Z}_2^s(\epsilon)$  is only an  $SO(3)$  action. We write  $\tilde{Z}_2(\epsilon)$  for the quotient by the  $SO(3)$  action on the frame.

The trivial strata  $T$  is given by  $\{\Lambda = 0\}$ . The inclusion  $\{\Lambda = 0\} \subseteq T$  follows because  $T$  is the set of fixed points of the  $SO(3)$  action on the frame. The reverse inclusion follows from the description in [BoM, equation 4.4.6] of the lower strata.

We are interested in  $Z_2^s(\epsilon)/SO(3)$ . A clever trick of Hattori's, from [Ha] allows us to eliminate the ADHM equation. He observes that the  $SU(2)$  Hopf fibration map gives us  $\text{Im}(\bar{\lambda}_1 \lambda_2)$ .

$$\begin{aligned} S^3 \setminus \mathbb{H}^2 &\xrightarrow{\sim} & \mathbb{C} \times \mathbb{R}^3 \\ [\lambda_1, \lambda_2] &\xrightarrow{\sim} & \left( \text{Re}(\bar{\lambda}_1 \lambda_2), \frac{|\lambda_1|^2 - |\lambda_2|^2}{2}, \text{Im}(\bar{\lambda}_1 \lambda_2) \right) \end{aligned}$$

The ADHM equation,  $\text{Im}(\bar{\lambda}_1 \lambda_2) = 2\text{Im}(\bar{c}a)$  allows us to forget the  $\mathbb{R}^3$  factor. Better still, this map is  $SO(2)$  equivariant, if  $SO(2)$  acts on  $\mathbb{C} \times \mathbb{R}^3$  by  $(z, x) \rightarrow (ze^{2i\theta}, x)$ . We have shown

**Lemma 3.3.1.**  $\tilde{Z}_2(\epsilon) \simeq \mathbb{H}^2 \times_{S^1} \mathbb{C}$ . Here  $S^1$  acts by  $R(\theta)$  on  $\mathbb{H}^2$  and by  $e^{i\theta}$  on  $\mathbb{C}$ . Under this equivalence, the trivial strata are given by  $T = \{[(a, c), z] : z = 0, \text{Im}(\bar{c}a) = 0\}$ . This homeomorphism is  $SO(4)$  equivariant if  $SO(4)$  acts on  $\mathbb{H}^2 \times_{S^1} \mathbb{C}$  by  $[h_1, h_2, z] \rightarrow [p_L h_1 \bar{p}_R, p_L h_2 \bar{p}_R, z]$ .

Analyzing the link of  $T$  is easier if we use a further decomposition of this space. We apply this same Hopf fibration to the  $SU(2)_L$  action and get a map

$$\begin{aligned} \mathbb{H}^2 \times_{S^1} \mathbb{C} &\xrightarrow{\pi} & \mathbb{C} \times \mathbb{R}^3 \times_{S^1} \mathbb{C} \\ [(a, c), z] &\xrightarrow{\sim} & \left( \text{Re}(\bar{c}a), \frac{|c|^2 - |a|^2}{2}, \text{Im}(\bar{c}a), z \right) \end{aligned}$$

In this description,  $T = \pi^{-1}([z_1, 0, 0])$ .

**Lemma 3.3.2.** *For the map  $\pi$  as above and co-ordinates  $(z_1, x, z_2)$  on  $\mathbb{C} \times \mathbb{R}^3 \times \mathbb{C}$*

$$\begin{aligned} T &= \pi^{-1}(\{x = z_2 = 0\}) = c(S^3 \times S^1)/S^1 \\ A &= \pi^{-1}(\{z_1 \neq 0\}) \simeq (S^3 \times \mathbb{R}^3 \times_{\mathbb{Z}/2} S^1 \times \mathbb{R}^+) \times_{S^1} \mathbb{C} \text{ where } S^1 \text{ acts on } S^1 \text{ and on } \mathbb{C} \text{ by } [\theta, z] \rightarrow [\theta + \phi, ze^{i\phi}]. \\ \mathbb{Z}/(2) &\text{ on the factors } S^3 \text{ and } S^1 \text{ by } (q, \theta) \rightarrow (-q, \theta + \pi). \\ B &= \pi^{-1}(\{z_1 = 0\}) \simeq c(S^3 \times S^2) \times_{S^1} \mathbb{C} \text{ where } S^1 \text{ acts by } (q, x, z) \rightarrow (qe^{-x\theta}, x, ze^{i\theta}) \end{aligned}$$

*Proof.* For  $[(a, c), z] \in T$ , such that  $\pi([(a, c), z]) = (z_1, 0, 0)$ , we have  $\bar{c}a = Re(z_1)$ . Thus  $(a, c) = q(r, s)$  for  $q \in S^3, r, s \in \mathbb{R}$ . The  $r, s$  give us the  $S^1$  and the cone parameter.

For  $[(a, c), z] \in B$ , such that  $\pi([(a, c), z]) = (0, x, z_2)$  we have  $\bar{c}a = x$ , a purely imaginary quaternion and thus  $x \in \mathbb{R}^3$  and  $|c| = |a|$ . Writing  $c = |c|q$  for  $q \in S^3$ , we then have  $a = \frac{qx}{|c|}$  so  $|a| = \frac{|x|}{|c|}$ . Thus  $|a| = |c| = \sqrt{|x|}$  and we have  $(a, c) = q(\frac{x}{\sqrt{|x|}}, \sqrt{|x|})$ . The  $SO(2)$  rotation action on  $(a, c)$  translates to

$$\begin{aligned} \left( \frac{qx}{\sqrt{|x|}}, q\sqrt{|x|} \right) &\rightarrow q \left( \frac{x}{\sqrt{|x|}} \cos(\theta) + \sqrt{|x|} \sin(\theta), -\frac{x}{\sqrt{|x|}} \sin(\theta) + \sqrt{|x|} \cos(\theta) \right) \\ &= q \left( \frac{x}{|x|} \sin(-\theta) + \cos(\theta) \right) \left( \frac{x}{\sqrt{|x|}}, \sqrt{|x|} \right) = qe^{-x\theta} \left( \frac{x}{\sqrt{|x|}}, \sqrt{|x|} \right) \end{aligned}$$

$B$  is then parametrized by, for  $q \in S^3, x \in \mathbb{R}^3$

$$\begin{aligned} c(S^3 \times S^2) \times_{S^1} \mathbb{C} &\xrightarrow{\phi_B} \mathbb{H}^2 \times_{S^1} \mathbb{C} \\ [(q, x), z] &\longrightarrow [q(\frac{x}{\sqrt{|x|}}, \sqrt{|x|}), z] \end{aligned}$$

For  $[(a, c), z] \in A$ , such that  $\pi([(a, c), z]) = (z_1, x, z_2)$ , we begin by assuming  $Im(z_1) = 0$  so that  $|a| = |c|$ . Then  $\bar{c}a = Re(z_1) + x$ . Let  $c = |c|q$  for  $q \in S^3$  and we have  $a = \frac{1}{|c|}q(Re(z_1) + x)$ . Then  $|a| = \frac{1}{|c|}\sqrt{|Re(z_1)|^2 + |x|^2}$  so  $|a| = |c| = \sqrt[4]{|Re(z_1)|^2 + |x|^2}$ . Thus for  $Im(z_1) = 0$ , we have

$$(a, c) = q \left( \frac{Re(z_1) + x}{\sqrt[4]{|Re(z_1)|^2 + |x|^2}}, \sqrt[4]{|Re(z_1)|^2 + |x|^2} \right)$$

We get rid of the assumption that  $Im(z_1) = 0$  by using the  $SO(2)$  equivariance of the Hopf fibration map, that is we rotate. However, since the rotation on  $(a, c)$  gives  $z_1 \rightarrow z_1 e^{2i\theta}$ , we must take an involution into account before parametrizing  $A$  by

$$\begin{aligned} S^3 \times \mathbb{R}^3 \times_{\mathbb{Z}/2} S^1 \times \mathbb{R}^+ \times_{S^1} \mathbb{C} &\xrightarrow{\phi_A} \mathbb{H}^2 \times_{S^1} \mathbb{C} \\ \phi_A([q, x, \theta, r, z]) &= q \left( \frac{r+x}{\sqrt[4]{|r|^2 + |x|^2}}, \sqrt[4]{|r|^2 + |x|^2} \right) \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \end{aligned}$$

The  $\mathbb{Z}/2$  involution is given by sending  $q \rightarrow -q$  and  $\theta \rightarrow \theta + \pi$ .

To complete our description of  $Z_2(\epsilon)$ , we must see how the spaces  $A$  and  $B$  fit together.

**Lemma 3.3.3.**  $\tilde{Z}_2(\epsilon)$  can be described as the mapping cone of the map

$$S^3 \times \mathbb{R}^3 \times_{\mathbb{Z}/(2)} S^1 \times_{S^1} \mathbb{C} \rightarrow c(S^3 \times S^2) \times_{S^1} \mathbb{C}$$

$$m([q, x, \theta, z] = [qe^{\frac{-x\theta}{|x|}}, \frac{x}{|x|}, |x|, z]$$

where  $|x|$  represents the cone parameter in  $c(S^3 \times S^2)$ .

*Proof.* This follows from  $\lim_{r \rightarrow 0} \phi_A([q, x, \theta, r, z]) = \phi_B([qe^{-x\theta}, x, z])$ .

We write  $\tilde{A}$  for the branched covering of  $A$  given by omitting the  $\mathbb{Z}/(2)$  action, so  $\tilde{A} = S^3 \times \mathbb{R}^3 \times \mathbb{C} \times \mathbb{R}^+$ . Lemma 3.3.2 shows us that the link of  $T$  in  $A$  is branch covered by the subset of  $\tilde{A}$  given by  $S^3 \times S^4 \times \mathbb{R}^+$  and the link in  $B$  is  $(S^3 \times S^2) * S^1 / S^1$ . Putting this together with Lemma 3.3.3, we have the following description of  $L$ .

**Lemma 3.3.4.** The link of  $T$  in  $Z_2(\epsilon)$ ,  $L$ , is degree four branch covered by the mapping cone of the natural map

$$m : S^3 \times (S^2 * S^1) \rightarrow (S^3 \times S^2) * S^1 / S^1.$$

The  $S^1$  action on  $(S^3 \times S^2) * S^1$  is given by

$$[q, x, \theta] \rightarrow [qe^{x\phi}, x, \theta + \phi].$$

This covering is  $SO(4)$  equivariant if  $[p_L, p_R] \in SO(4)$  acts by

$$[q, x, z] \rightarrow [p_L q \bar{p}_R, p_R x \bar{p}_R, z]$$

where  $q \in S^3, x \in \mathbb{R}^3, z \in \mathbb{C}$  on  $B$  and on  $A$  by, for  $\theta \in S^1, r \in \mathbb{R}^+$ :

$$[q, x, \theta, r, z] \rightarrow [p_L q \bar{p}_R, p_R x \bar{p}_R, \theta, r, z].$$

*Proof.* This follows from Lemma 3.3.3. The equivariance follows by noting that the parametrizing maps  $\phi_A$  and  $\phi_B$  are equivariant with respect to these  $SO(4)$  actions.

We write  $\tilde{L}$  for the branch cover of the link  $L$  described in Lemma 3.3.4. This lemma shows us  $\partial L$  is branch covered by  $S^3 \times S^4 \subset \tilde{L} \cap \tilde{A}$ . The compactification is  $L \cup C$  is given by gluing on cones on the  $SO(3)_L$  orbits. This action is trivial on  $S^4$  and the standard action on  $S^3$ . Thus, we have

**Lemma 3.3.5.** The compactification  $L \cup C$  is degree four branch covered by

$$(D^4 \times S^4) \cup (S^3 \times S^4 \times \mathbb{R}^+) \cup ((S^3 \times S^2) * S^1 / S^1).$$

This covering is  $SO(4)$  equivariant if the action is as in Lemma 3.3.4 on the second and third open sets and  $[q, x, z] \rightarrow [p_L q \bar{p}_R, p_R x \bar{p}_R, z]$  on the first open set, for  $q \in S^3, x \in S^2, z \in \mathbb{C}$ .

Our compactification thus decomposes naturally into the mapping cone  $\tilde{L}$  and the cap  $D^4 \times S^4$ . The cohomology of the latter is obvious, that of the former is computed by noting that it retracts onto the image  $B \cap L$  and using the following lemma.

**Lemma 3.3.6.**  $H^*(B \cap L) \simeq H^*(S^3 \times S^2 \times_{S^1} (\mathbb{C}, S^1))$  for  $* > 0$ .

*Proof.* The Meyer-Vietoris sequence for the decomposition

$$B \cap L = (S^3 \times S^2 \times_{S^1} \mathbb{C}) \cup (c(S^3 \times S^2) \times_{S^1} S^1)$$

is the same as that of the pair  $S^3 \times S^2 \times_{S^1} (\mathbb{C}, S^1)$  because the second of the two open sets retracts to a point.

Another useful piece of information is the following Euler class computation.

**Lemma 3.3.7.** ([Ha]) *The quotient map of the  $S^1$  action on  $S^3 \times S^2$  given by  $(q, x) \rightarrow (qe^{-x\theta}, x)$  can be written  $(q, x) \rightarrow (qx\bar{q}, x)$ . The Chern class of the bundle  $S^3 \times S^2 \rightarrow S^2 \times S^2$  given by  $[q, x] \rightarrow [qx\bar{q}, x]$  is  $\eta_L - \eta_R$ , where  $\eta_L$  is the Poincare dual of  $\{p\} \times S^2$  and  $\eta_R$  that of  $S^2 \times \{p\}$  in  $S^2 \times S^2$ .*

*Proof.* If we restrict the bundle to  $S^2 \times \{i\}$ , we have the projection map  $[q, i] \rightarrow [qi\bar{q}, i]$  which is the standard right  $S^1$  action on  $S^3$ . On the second  $S^2$ , that is  $\{i\} \times S^2$ , the bundle is  $\{[q, x] : qx\bar{q} = i\} = \{[q, x] : \bar{q}iq = x\}$  which is given by a left  $S^1$  action on  $S^3$ . Thus the co-efficient of the generator of the right  $S^2$  in this  $c_1$  is  $-1$ .

**Definition 3.3.8.** *Let  $T \in H^2(B \cap L)$  be the class corresponding, under the isomorphism of Lemma 3.3.6 to the Thom class.*

Thus by Lemma 3.3.7, if  $i : S^2 \times S^2 \rightarrow B \cap L$  is the inclusion, we have  $i^*T = \eta_L - \eta_R$ . To compute the cohomology ring of the compactification, we need to know  $m^* : H^*(B \cap L) \rightarrow H^*(S^3 \times S^4)$ .

**Lemma 3.3.9.**  $m^*(T\eta_R) = m^*(T\eta_L) = PD[S^3 \times \{p\}]$  for  $p \in S^4$ .

*Proof.* The map  $m$  respects the decompositions  $S^3 \times S^4 = (S^3 \times S^2 \times \mathbb{C}) \cup (S^3 \times \mathbb{R}^3 \times S^1)$  and  $B \cap L = (S^3 \times S^2 \times_{S^1} \mathbb{C}) \cup (c(S^3 \times S^2) \times_{S^1} S^1)$ . The four dimensional cohomology of both  $S^3 \times S^4$  and  $B \cap L$  is generated by the image of the cohomology of the pairs,  $S^3 \times S^2 \times (\mathbb{C}, S^1)$  and  $S^3 \times S^2 \times_{S^1} (\mathbb{C}, S^1)$ . Thus, it suffices to discuss  $m^* : H^4(S^3 \times S^2 \times_{S^1} (\mathbb{C}, S^1)) \rightarrow H^4(S^3 \times S^2 \times (\mathbb{C}, S^1))$ . We look at the exact sequences of the pairs, writing  $m_0$  for  $m$  restricted to the boundaries,

$$\begin{array}{ccccc} S^3 \times S^2 \times S^1 & \xrightarrow{i'} & S^3 \times S^2 \times \mathbb{C} & \xrightarrow{j'} & S^3 \times S^2 \times (\mathbb{C}, S^1) \\ m_0 \downarrow & & m \downarrow & & m \downarrow \\ S^3 \times S^2 \times_{S^1} & \xrightarrow{i} & S^3 \times S^2 \times_{S^1} \mathbb{C} & \xrightarrow{j} & S^3 \times S^2 \times_{S^1} (\mathbb{C}, S^1) \end{array}$$

Let  $\eta'$  be the Poincare dual of  $S^3 \times \{i\} \times \mathbb{C}$  in the upper sequence, for  $i \in S^2$ , let  $T'$  be the dual of  $S^3 \times S^2 \times \{0\}$  in the relative cohomology of the upper sequence. Let  $T, \eta_L, \eta_R$  be as definition 3.3.8 and Lemma 3.3.7. Because  $T$  is dual to  $S^3 \times S^2 \times_{S^1} \{0\}$ ,  $m^*T$  is dual to  $S^3 \times S^2 \times \{0\}$ , so  $m^*T = T'$ . We see  $m^*(\eta_L - \eta_R) = m^*j^*T = j^*T' = 0$ . Finally, we see  $T\eta_R$  is dual to  $S^3 \times \{i\} \times_{S^1} \{0\}$  for  $i \in S^2$ , so  $m^*(T\eta_R)$  is dual to  $S^3 \times \{i\} \times \{0\}$  so  $m^*(T\eta_R) = T'\eta'$ .

Now that we have described the cohomology of  $\tilde{L} \cup C$ , we wish to extend this description to an equivariant one. It seems easier to work with  $\tilde{G} = SU(2) \times SU(2)$  equivariant cohomology than with  $SO(4)$ . Working with rational coefficients, no information is lost, as evidenced by this lemma.

**Lemma 3.3.10.** [HH] *Let  $c_L, c_R$  be the second Chern classes of the bundles  $E\tilde{G}/SU(2)_R \rightarrow B\tilde{G}$  and  $E\tilde{G}/SU(2)_L \rightarrow B\tilde{G}$  respectively, then  $H^*(B\tilde{G}; \mathbb{Q}) = \mathbb{Q}[c_L, c_R]$ .*

*If  $s : B\tilde{G} \rightarrow BSO(4)$  is the classifying map for the natural  $SO(4)$  bundle over  $B\tilde{G}$  and  $e, p_1 \in H^4(BSO(4))$  are the universal Euler class and the universal Pontrjagin class, then  $s^*(p_1 + 2e) = -4c_R$  and  $s^*(p_1 - 2e) = -4c_L$ .*

We now describe the equivariant cohomology of the cap  $D^4 \times S^4$  and calculate the restriction of the equivariant extension of  $\phi$  to this set.

**Lemma 3.3.11.** *If  $H \in H_{\tilde{G}}^4(D^4 \times S^4)$  is the generator of the cohomology of the fiber of*

$$\pi_{\tilde{G}} : E\tilde{G} \times_{\tilde{G}} (D^4 \times S^4) \rightarrow B\tilde{G},$$

*then*

$$H_{\tilde{G}}^*(D^4 \times S^4) \simeq H^*(B\tilde{G})[H]/(H^2 + 2H\pi_{\tilde{G}}^*c_R + \pi_{\tilde{G}}^*c_R).$$

*Restricted to  $D^4 \times S^4$ , the equivariant extension of  $\varphi$  equals  $-4H - 4\pi_{\tilde{G}}^*c_R$ .*

*Proof.* The assertion about the structure of the equivariant cohomology ring follows by noting that  $D^4 \times S^4$  retracts equivariantly onto  $S^4$  and the  $\tilde{G}$  action on  $S^4$ ,  $[x, z] \rightarrow [p_R x \bar{p}_R, z]$  for  $x \in \mathbb{R}^3, z \in \mathbb{C}$  is equivalent to the action  $[h_1 \bar{p}_R, h_2 \bar{p}_R]$  on  $\mathbb{HP}^1$  ( $h_i \in \mathbb{H}$ ). Our homotopy quotient retracts onto the quaternionic projectivization of  $BSU(2)_L \times ESU(2)_R \times_{SU(2)_R} \mathbb{H}^2$  where the cohomology ring is as described.

The four dimensional cohomology of this homotopy quotient, then has three generators  $H, c_R, c_L$ . We calculate the co-efficient of  $H$  in  $\varphi$  by restricting to a single fiber. We calculate the co-efficients of  $c_R, c_L$  by restricting to the section of  $\pi_{\tilde{G}}$  given by  $E\tilde{G} \times_{\tilde{G}} [1, 0, e^0]$ , for  $1 \in S^3, 0 \in \mathbb{R}^3, e^0 \in \mathbb{C}$ .

Restricted to a single fiber, we calculate the bundle as follows. The Hopf fibration in Lemma 3.3.1, and thus the fibration associated to the  $SO(3)$  framing action, can be seen as the restriction of  $\mathbb{H}^2 \times_{S^1} \mathbb{H}^2 \rightarrow \mathbb{H}^2 \times_{S^1} B^5$  to the image of

$$\begin{aligned} \mathbb{H}^2 \times_{S^1} \mathbb{C} &\longrightarrow \mathbb{H}^2 \times_{S^1} B^5 \\ [(a, c), z] &\longrightarrow [(a, c), (\frac{1}{2} \operatorname{Im}(\bar{c}a), z)] \end{aligned}$$

Of course, the Hopf fibration is not a fibration over  $0 \in B^5$ , but the intersection of the image of  $\mathbb{H}^2 \times_{S^1} \mathbb{C}$  with  $\mathbb{H}^2 \times \{0\}$  is simply  $T$ . Composing the above inclusion with  $\phi_A$  we get an inclusion  $S^3 \times S^4 \rightarrow \mathbb{H}^2 \times_{S^1} B^5$  by

$$\begin{aligned} S^3 \times B^5 &\longrightarrow \mathbb{H}^2 \times_{S^1} B^5 \\ [q, (x, z)] &\longrightarrow \left[ q \left( \frac{1+x}{\sqrt[4]{1+|x|^2}}, \sqrt[4]{|1+|x|^2} \right), (x, z) \right] \end{aligned}$$

Here  $x \in \mathbb{R}^3, z \in \mathbb{C}$ . The Hopf fibration then pulls back to the Hopf fibration on the factor  $S^4$ , so the associated  $SO(3)$  bundle has  $p_1$  equal to  $-4$  times the generator of  $H^4(S^4)$ .

Then, when we restrict to the section,  $E\tilde{G} \times_{\tilde{G}} (1, 0, e^0)$ , we see  $(\lambda_1, \lambda_2)$  must satisfy  $\bar{\lambda}_1 \lambda_2 = 1, |\lambda_1| = |\lambda_2|$  and can thus be written  $(\lambda_1, \lambda_2) = (q, q)$  for some  $q \in S^3$ . The bundle is then  $SO(3)$  bundle associated to

$$E\tilde{G} \times_{\tilde{G}} S^3 \rightarrow B\tilde{G}$$

where  $\tilde{G}$  acts on  $S^3$  by  $q \rightarrow q\bar{p}_R$ . This has  $p_1 = -4c_R$ .

The mapping cone of Lemma 3.3.4 retracts equivariantly onto its image and we can repeat the computations of Lemma 3.3.6, equivariantly.

**Lemma 3.3.12.** *If  $\eta_L, \eta_R \in H_{\tilde{G}}^2(S^3 \times S^2/S^1)$  are the generators of the cohomology of the fiber of  $E\tilde{G} \times_{\tilde{G}} (S^3 \times S^2)/S^1 \rightarrow B\tilde{G}$  where  $\tilde{G}$  acts on  $S^3 \times S^2$  and  $(S^3 \times S^2)/S^1$  as in Lemma 3.3.4, then*

$$\begin{aligned} H_{\tilde{G}}^*(S^3 \times S^2/S^1) &\simeq H^*(B\tilde{G})[\eta_L, \eta_R]/(\eta_L^2 + \pi^*c_L, \eta_R^2 + \pi^*c_R) \simeq \mathbb{Q}[\eta_L, \eta_R], \\ H_{\tilde{G}}^*(S^3 \times S^2) &\simeq H^*(B\tilde{G})[\eta_L, \eta_R]/(\eta_L - \eta_R, \eta_L^2 + \pi^*c_L, \eta_R^2 + \pi^*c_R) \simeq \mathbb{Q}[y] \end{aligned}$$

where  $y$  is a two dimensional cohomology class.

$$H_{\tilde{G}}^*(B \cap L) \simeq \operatorname{Ker} \left[ H_{\tilde{G}}^*(S^3 \times S^2/S^1) \oplus H^*(B\tilde{G}) \rightarrow H_{\tilde{G}}^*(S^3 \times S^2) \right].$$

*Proof.* The first assertion follows by noting that the projection

$$S^3 \times S^2 \rightarrow S^3 \times S^2 / S^1$$

where  $S^1$  acts by  $[qe^{-x\theta}, x]$  is given by  $(q, x) \rightarrow (qx\bar{q}, x)$ . Thus the  $\tilde{G}$  action on  $S^3 \times S^2 / S^1 = S^2 \times S^2$  is given by  $(x_{1,2}) \rightarrow (p_L x \bar{p}_L, p_R x \bar{p}_R)$ . The homotopy quotient of this action on  $S^2 \times S^2$  is the product of two copies of  $ESU(2)/S^1$  which gives the first assertion.

The second assertion follows from noting that the homotopy quotient of  $S^3 \times S^2$  is an  $S^1$  line bundle over that of  $S^3 \times S^2 / S^1$  and the Euler class of this bundle is given by Lemma 3.3.7. Note that  $H_G^2(S^2 \times S^2)$  is generated on a single fiber, so to calculate the Euler class of a line bundle, it suffices to restrict our attention to a single fiber and the non-equivariant computation holds for the homotopy quotient.

The third assertion follows from the Meyer-Vietoris sequence and noting that the cohomology of the intersection,  $H_G^*(S^3 \times S^2)$  vanishes in odd dimensions to give injectivity.

By virtue of this injectivity, to define global cohomology classes on the  $\tilde{L}$ , it suffices to give their images in homotopy quotients of  $S^3 \times S^2 \times_{S^1} \mathbb{C}$  and  $c(S^3 \times S^2) \times_{S^1} S^1$ .

**Definition 3.3.13.** *The equivariant cohomology classes  $T, K_L, K_R$  are defined by*

$$\begin{aligned} T &\rightarrow (\eta_L - \eta_R, 0) \in H_G^2(S^3 \times S^2 / S^1) \oplus H^2(B\tilde{G}) \\ K_L &\rightarrow (c_L - \eta_L \eta_R, 0) \in H_G^4(S^3 \times S^2 / S^1) \oplus H^4(B\tilde{G}) \\ K_R &\rightarrow (\eta_L \eta_R - c_R, 0) \in H_G^4(S^3 \times S^2 / S^1) \oplus H^4(B\tilde{G}) \end{aligned}$$

A note on the origins of  $K_L$ : fiberwise, this class comes from  $T\eta_L$ . We compute some useful relations among these classes.

**Lemma 3.3.14.**

$$\begin{aligned} T^2 &\rightarrow (-c_L - 2\eta_R \eta_L - c_R, 0) \\ K_L^2 &= -T^2 c_L \quad K_R^2 = -T^2 c_R \end{aligned}$$

*Proof.* Just observe these are equal when restricted to both open sets.

**Lemma 3.3.15.** ([Ha]) *Restricted to  $\tilde{L}$ ,  $\varphi = T^2 - 4K_R - 4c_R$ .*

*Proof.* To compute, we restrict to the two open sets given by the homotopy quotients of  $m^{-1}(S^3 \times S^2 \times_{S^1} \mathbb{C})$  and  $m^{-1}(c(S^3 \times S^2) \times_{S^1} S^1)$ .

The set  $S^2 \times S^2 \subset L \cap B$ , can be written as  $[q, x, 0], q \in S^3, x \in S^2$ . The parametrizing map  $\phi_B$  takes these points to  $[(qx, q), 0] \in \mathbb{H}^2 \times_{S^1} \mathbb{C}$ . Over this set,  $(\lambda_1, \lambda_2)$  satisfy  $\bar{\lambda}_1 \lambda_2 = x$  and  $|\lambda_1| = |\lambda_2|$ . We have  $(\lambda_1, \lambda_2) = q'(1, x)$ . The  $SO(2)$  action on these  $\lambda_i$  is given by  $q'(1, x) \rightarrow q'e^{x\theta}(1, x)$ . To get the  $SO(3)$  bundle, we map  $q' \rightarrow Ad(q')$ . The frame bundle over  $S^2 \times S^2$  is then

$$(S^3 \times S^2) \times_{S^1} SO(3) \rightarrow (S^3 \times S^2) / S^1$$

where  $S^1$  acts by  $[q_1, x, Ad(q_2)] \rightarrow [q_1 e^{-2x\theta}, x, Ad(q_2 e^{x\theta})]$ . If we denote the  $S^1$  bundle  $S^3 \times S^2$  over  $S^2 \times S^2$  given in Lemma 3.3.6 by  $V$ , then Hattori shows that our  $SO(3)$  bundle is the  $SO(3)$  extension of the  $S^1$  bundle given by tensoring  $V^{-1}$  with the  $S^1$  bundle given by pulling back  $SO(3) \rightarrow S^2$  ( $c_1 = 2$ ) from the right  $S^2$ . The argument for this goes as follows. This tensor product of  $S^1$  bundles,  $V^{-1} \otimes \pi_R^* SO(3)$ , can be described as the quotient of  $S^3 \times SO(3)$  by the  $S^1$  action  $[q_1, Ad(q_2)] \sim [q_1 e^{2q_2 i \bar{q}_2 \theta}, Ad(q_2 e^{i\theta})]$ . The projection of this space to  $S^2 \times S^2$  is given by  $[q_1, Ad(q_2)] \rightarrow [q_1 q_2 i \bar{q}_2 \bar{q}_1, q_2 i \bar{q}_2]$ . The  $S^1$  action on this  $S^1$  bundle is then

given by  $[q_1, Ad(q_2)] \rightarrow [q_1 e^{-x\theta}, Ad(q_2)] \sim [q_1, Ad(q_2)e^{i\theta}]$ . The  $SO(3)$  extension of  $V^{-1} \otimes \pi_R^* SO(3)$  is given by  $[q_1, Ad(q_2), Ad(q_3)] \sim [q_1 e^{-2x\theta}, Ad(q_2), Ad(q_3e^{i\theta})] \sim [q_1, Ad(q_2e^{i\theta}), Ad(q_3e^{i\theta})]$ . Finally, Hattori gives the map  $V^{-1} \otimes \pi_R^* SO(3) \times SO(3)/S^1 \rightarrow S^3 \times S^2 \times SO(3)/S^1$  by:  $[q_1, Ad(q_2), Ad(q_3)] \rightarrow [q_1, q_2i\bar{q}_2, Ad(q_3\bar{q}_2)]$ . One checks that this map respects the  $S^1$  action and  $\varphi$  and  $(\eta_L - 3\eta_R)^2$  are equal on the first open set. We see  $(T^2 - 4K_R - 4c_R)$  also maps to  $(\eta_L - 3\eta_R)^2$  when restricted this set (use  $T - 2\eta_R = \eta_L - 3\eta_R$ ).

The fiberwise contractible open set  $E\tilde{G} \times_{\tilde{G}} c(S^3 \times S^2) \times_{S^1}$  is easier. The space  $c(S^3 \times S^2) \times_{S^1} S^1$  retracts  $\tilde{G}$  equivariantly onto the cone point. Taking the point in  $S^1$  to be 1, so  $\bar{\lambda}_1\lambda_2 = 1$  and  $\lambda_1 = \lambda_2 \in S^3$ , the bundle over the cone point is simply  $E\tilde{G} \times_{\tilde{G}} S^3$ , where  $\tilde{G}$  acts on  $S^3$  by  $q \rightarrow p_R q$ . Again  $T - 4K_R - 4c_R$  has the same image under restriction to this open set.

We have calculated the restriction of  $\varphi$  to two open sets covering the compactification. We now observe that this suffices to give the global cohomology class. In general, the possible global extensions of these two restrictions would be an affine space, with underlying vector space given by the image of the cohomology of the intersection of the two open sets under the boundary map of the Meyer-Vietoris sequence. We see, however, that our insistence upon working equivariantly eliminates this ambiguity.

**Lemma 3.3.16.**  $H_{\tilde{G}}^*(S^3 \times S^4)$  vanishes in odd dimensions.

*Proof.* We use the Meyer-Vietoris sequence for the decomposition

$$E\tilde{G} \times_{\tilde{G}} (S^3 \times S^4) = \left( E\tilde{G} \times_{\tilde{G}} (S^3 \times S^2 \times \mathbb{C}) \right) \cup \left( E\tilde{G} \times_{\tilde{G}} (S^3 \times \mathbb{R}^3 \times S^1) \right).$$

The cohomology ring  $H_{\tilde{G}}^*(S^3 \times S^2 \times \mathbb{C})$  has been calculated in Lemma 3.3.9 and can be seen to be a polynomial ring with one two dimensional generator,  $\mathbb{Q}[y]$ . The second open set retracts onto  $(E\tilde{G} \times_{\tilde{G}} S^3) \times S^1$  which has cohomology ring  $H^*(B\tilde{G})/(c_L + c_R) \otimes \mathbb{Q}[\theta]/(\theta^2)$ , where  $\theta$  is one dimensional. The intersection of the two open sets is  $E\tilde{G} \times_{\tilde{G}} (S^3 \times S^2) \times S^1$  which has cohomology ring  $\mathbb{Q}[y'] \otimes \mathbb{Q}[\theta']/((\theta')^2)$  where again  $\theta'$  is one dimensional. We see that the one dimensional cohomology class  $\theta$  maps onto the one dimensional cohomology of the intersection,  $\theta'$ , and  $\mathbb{Q}[y]$  maps onto  $\mathbb{Q}[y']$ , so none of the odd dimensional cohomology is global.

Thus, our restriction of the equivariant extension of  $\varphi$  to the two open sets  $E\tilde{G} \times_{\tilde{G}} (D^4 \times S^4)$  and  $E\tilde{G} \times_{\tilde{G}} \tilde{L}$  have unique lifts to cohomology classes with compact support. We define the relative pushforward maps by

$$\begin{aligned} p_1 &: E\tilde{G} \times_{\tilde{G}} ((D^4, S^3) \times S^4) \rightarrow B\tilde{G} \\ p_2 &: E\tilde{G} \times_{\tilde{G}} (B \cap L, S^3 \times S^4) \rightarrow B\tilde{G}. \end{aligned}$$

We then calculate

$$p_*(\varphi^r) = (p_1)_*((-4H - 4c_R)^r) + (p_2)_*(T^2 - 4K_R - 4c_R)^r.$$

We note that, because the dimension of the fundamental class of the fiber is 8, we can restrict to  $E\tilde{G}_{r-8} = \pi_{\tilde{G}}^{-1} B\tilde{G}_{r-8}$ , where  $B\tilde{G}_{r-8}$  is the  $r-8$  skeleton of some CW decomposition of  $B\tilde{G}$ . We see the first term will vanish. We use this restriction property and the relation  $H^2 + c_R H + c_R^2 = 0$  to calculate.

$$\begin{aligned} (p_1)_*((-4)^2(H^2 + 2Hc_R + c_R^2)) &= 16(p_1)_*(H^2) = 16(p_1)_*(-2c_R H + c_R^2) = 0 \\ (p_1)_*((-4)^3(H^3 + 3H^2c_R + 3Hc_R^2 + c_R^3)) &= -64(p_1)_*(H(-2c_R H - c_R^2) + 3c_R(-2c_R H + c_R^2)) = 0. \end{aligned}$$

To compute the second term, involving  $(p_2)_*$ , we need the following lemma.

**Lemma 3.3.17.** *Let  $i$  be the inclusion of  $E\tilde{G} \times_{\tilde{G}} (S^2 \times S^2)$  into the homotopy quotient of the mapping cone of Lemma. Let  $p_3 : E\tilde{G} \times_{\tilde{G}} (S^2 \times S^2) \rightarrow B\tilde{G}$  be the projection. Then for any  $x \in H_{\tilde{G}}^*(B \cap L, S^3 \times S^4)$ , we have*

$$(p_2)_*(T^2 x) = (p_3)_*(i^* x).$$

*The pushforward map  $(p_3)_*$  is division by  $\eta_L \eta_R$ .*

*Proof.* This is equivalent to the assertion that  $T^2$  is the Poincare dual of the space  $E\tilde{G}_r \times_{\tilde{G}} (S^2 \times S^2)$  for any finite  $r$ . A neighborhood of this space in the homotopy quotient of the mapping cone is given by

$$E\tilde{G}_r \times_{\tilde{G}} (S^3 \times S^2 \times_{S^1} \mathbb{C}^2)$$

where one factor of  $\mathbb{C}$  comes from the inclusion of  $S^2 \times S^2$  into  $B \cap L$  and the other factor arises from the mapping cone. One can see this by noting that restricted to the open set of  $B \cap L$  given by  $S^3 \times S^2 \times_{S^1} \mathbb{C}$ , the map  $m$  is given by

$$m : S^3 \times S^2 \times S^1 \times_{S^1} \mathbb{C} \rightarrow S^3 \times S^2 \times_{S^1} \mathbb{C}.$$

The mapping cone of this restriction of  $m$  is then  $S^3 \times S^2 \times_{S^1} \mathbb{C}^2$ . The class  $T^2$  restricts to this open set of the mapping cone to be the Thom class of the normal bundle of  $E\tilde{G}_r \times_{\tilde{G}} (S^2 \times S^2)$  and vanishes on the complement.

We are now ready to perform our computation. We use the restriction property: for calculating  $(p_2)_*(\phi^r)$  we may restrict to  $E\tilde{G}_{4r-8} \times_{\tilde{G}} \tilde{L}$  so any product  $c_R^i c_L^j$  will vanish if  $i + j \geq r - 2$ . We also recall from the definition of  $T, K_R$  that  $i^* T = \eta_L - \eta_R$  and  $i^* K_R = \eta_L \eta_R + c_R$ , where  $i$  is as in Lemma 3.3.17. Finally, we employ the cohomology relations of Lemma 3.3.14. We compute:

$$\begin{aligned} (p_2)_* ((T^2 - 4K_R - 4c_R)^2) &= (p_2)_* (T^4 - 8T^2 K_R + 16K_R^2) = (p_2)_* (T^2 (T^2 - 8K_R - 16c_R)) \\ &= (p_2)_* (T^2 (T^2 - 8K_R)) = (p_3)_* (i^* (T^2 - 8K_R)) = (p_3)_* (-c_L - 2\eta_L \eta_R - c_R - 8\eta_L \eta_R - 8c_R) \\ &= -10 \\ (p_2)_* ((T^2 - 4K_R - 4c_R)^3) &= (p_2)_* ((T^2 - 4K_R)^3 - 12(T^2 - 4K_R)^2 c_R) \\ &= (p_2)_* (T^6 - 12T^4 K_R + 48T^2 K_R^2 - 64K_R^3 - 12T^4 c_R + 96T^2 K_R c_R - 12(16)K_R^2 c_R) \\ &= (p_2)_* (T^2 (T^4 - 12T^2 K_R + 48K_R^2 + 64K_R c_R - 12T^2 c_R + 96K_R c_R - 12(16)c_R^2)) \\ &= (p_3)_* i^* ((\eta_L - \eta_R)^4 - 12(\eta_L - \eta_R)^2(\eta_L \eta_R + c_R) + 48(\eta_L \eta_R + c_R)^2 + 64(\eta_L \eta_R + c_R)c_R \\ &\quad - 12(\eta_L - \eta_R)^2 c_R + 96(\eta_L \eta_R + c_R)c_R) \\ &= (p_3)_* (c_L^4 + 4c_L \eta_L \eta_R + 6c_R c_L + 4\eta_L \eta_R c_R + c_R^2 - 12(-c_L \eta_L \eta_R - 3c_R \eta_L \eta_R) \\ &\quad + 48(c_L c_R + 2\eta_L \eta_R c_R + c_R^2) + 64(\eta_L \eta_R c_R + c_R^2) \\ &\quad - 12(-c_L c_R - 2\eta_L \eta_R c_R - c_R^2) + 96(\eta_L \eta_R c_R + c_R^2)) \\ &= (4 + 12)c_L + (4 + 36 + 96 + 64 + 24 + 96)c_R = 16c_L + 320c_R \end{aligned}$$

Dividing by the degree of the branched covering, 4, have performed the following computation.

**Lemma 3.3.18.**

$$p_*(\phi^2) = -\frac{5}{2}, \quad p_*(\phi^3) = -s^*(p_1 - 2e) - 20s^*(p_1 + 2e).$$

Pulling back to  $X$ , where  $b^1(X) = 0, b^+(X) = 1$ , we can see

$$\begin{aligned} p_+(X) &= p_1(X) + 2e(X) = 3(b^+ - b^-) + 4 + 2b^+ + 2b^- = 9 - b^-, \\ p_-(X) &= p_1(X) - 2e(X) = 3(b^+ - b^-) - 4 - 2b^+ - 2b^- = -3 - 5b^-. \end{aligned}$$

From this, we can derive  $p_-(X) = -48 + 5p_+(X)$  This gives us

**Proposition 3.3.19.** *For the fibration  $p : Fr(X) \times_{SO(4)} (L \cup C_2) \rightarrow X$  where  $X$  satisfies  $b^1(X) = 0, b^+(X) = 1$ , if we denote  $p_+ = p_1(X) + 2e(X)$ , we have*

$$p_*(\varphi)^2 = -\frac{5}{2} \quad p_*(\varphi^3) = 48 - 25p_+$$

### 3.4: The Error Term

We covered a neighborhood of the family of reducibles  $[\alpha] \times \Sigma^2(X)$  with two open sets  $\nu_{\Sigma_{(1,1)}}$  and  $\nu_{\Sigma_2}$ . We compactified  $\nu_{\Sigma_{(1,1)}}$  by extending the gluing data bundles over the diagonal, creating a new space  $\nu_{\Sigma_{(1,1)}} \cup C_1$ , and compactified  $\nu_{\Sigma_2}$  by gluing on cones on  $SO(3)_L$  orbits, creating the space  $\nu_{\Sigma_2} \cup C_2$ . Although the caps do not represent actual connections, we persist in using the term reducible to describe those points in the compactified spaces of gluing data where the  $\Gamma_\alpha$  action is not free. We write  $(\nu_{\Sigma_{(1,1)}} \cup C_1)^\circ$  and  $(\nu_{\Sigma_2} \cup C_2)^\circ$  for the links of the reducibles in these space. The computation can then be expressed as:

$$(3.4.1) \quad \langle \bar{\mu}(z), [\partial\nu_\alpha] \rangle = \langle \bar{\mu}(z), [(\nu_{\Sigma_2} \cup C_2)^\circ] \rangle + \langle \bar{\mu}(z), [(\nu_{\Sigma_2} \cup C_2)^\circ] \rangle - \langle \bar{\mu}(z), [(C_1 \cup C_2)^\circ] \rangle.$$

The first two terms on the right hand side were computed in sections 3.1 and 3.2. Our objective in this section is to show that the final term of this equation vanishes.

The cap  $C_1$  is obtained by extending the gluing data bundles over the diagonal  $\Delta \subset \Sigma^2(X)$ . We then have a natural homeomorphism:

$$k : \mathbb{C}^N \times_{\Gamma_\alpha} (Q_\alpha \times_X Fr(X) \times_{SO(4)} (D^4 \times c(SO(3)) \times c(SO(3))) / \mathbb{Z}/(2)) / S^1 \rightarrow C_1.$$

Here the  $S^1$  acts on  $Q_\alpha$  and  $c(SO(3)) \times c(SO(3))$  while  $\mathbb{Z}/(2)$  acts by  $-1$  on  $D^4$  and by permuting the factors of  $c(SO(3))$ . The  $SO(4)$  action is the standard action on  $D^4$  and  $SO(3)_R$  on each factor of  $c(SO(3))$ . The homeomorphism  $k$  is given by noting that a neighborhood of  $\Delta \subset \Sigma^2(X)$  is described by  $Fr(X) \times_{SO(4)} D^4 / (\mathbb{Z}/(2))$ . The frames of  $Q_\alpha$  and  $Fr(X)$  are parallel translated to frames at the point in  $\Sigma^2(X)$  represented by the point of  $Fr(X) \times_{SO(4)} D^4 / (\mathbb{Z}/(2))$ . In the case  $X = \mathbb{R}^4$ , we could think of this as parallel translating the frames to the points  $y$  and  $-y$ , where  $y$  is the point in  $Fr(X) \times_{SO(4)} D^4 / (\mathbb{Z}/(2))$ . The product  $c(SO(3)) \times c(SO(3))$  represents two charge one framed instantons on  $S^4$ . We write  $C_1^s(x)$  for the cap on a fiber:  $(D^4 \times c(SO(3)) \times c(SO(3))) / \mathbb{Z}/(2)$ . The reducibles in this cap are given by

$$\{0\} \times_{\Gamma_\alpha} (Q_\alpha \times_X Fr(X) \times_{SO(4)} (D^4 \times c \times c) / \mathbb{Z}/(2)) / S^1$$

where  $c \in c(SO(3))$  is the cone point.

The cap  $C_2$  is described by attaching cones on  $SO(3)_L$  orbits to  $\partial Z_2^s(\epsilon)$ . We write  $C_2^s(x)$  for the space obtained by these cones. Because the rotation action on  $Z_2^s(\epsilon)$  respects the stratification, the stratification extends over this cap. The cap  $C_2$  is then given by

$$\mathbb{C}^N \times_{\Gamma_\alpha} (Q_\alpha \times_X Fr(X) \times_{SO(4)} C_2^s(x)) / S^1$$

If we write  $\bar{T}$  for the cones on  $SO(3)_L$  orbits in  $Z_2^s(\epsilon)$  where the background connection is trivial, the reducibles in this cap will be given by

$$\{0\} \times_{\Gamma_\alpha} (Q_\alpha \times_X Fr(X) \times_{SO(4)} \bar{T}) / S^1.$$

We distinguish between the splicing map, where trivializations and cut-off functions are used to splice connections on  $S^4$  to the background connections on  $Q_\alpha \times_{S^1} SO(3)$ , and the gluing map, where a perturbation is added to the spliced connection to make it anti-self dual. We write  $\gamma'_{(1,1)}$  for the splicing map associated to the gluing data bundle of the off-diagonal and  $\gamma_{(1,1)}$  for the gluing map. Similarly, we write  $\gamma'_2$  for the splicing map of a charge two  $S^4$  instanton along the diagonal and  $\gamma_2$  for the gluing map. Then  $\gamma_2^{-1} \circ \gamma_{(1,1)}(\nu_{\Sigma_{(1,1)}}) \subset \nu_{\Sigma_2}$  has a boundary with two components, one from the boundary of  $\nu_{\Sigma_2}$  and one from the image of the boundary of  $\nu_{\Sigma_{(1,1)}}$ . Because  $\nu_{\Sigma_{(1,1)}}$  and  $\nu_{\Sigma_2}$  give an open cover, these two boundaries are disjoint. If we attach  $C_1$  and  $C_2$  to the appropriate boundaries, we obtain a space we call  $C_2 \cup_{\gamma_2^{-1} \circ \gamma_{(1,1)}} C_1$ . The gluing maps respect the reducibles, so it makes sense to write the link of the reducibles in this space as  $(C_2 \cup_{\gamma_2^{-1} \circ \gamma_{(1,1)}} C_1)^\circ$ . Our error term is then

$$\langle \bar{\mu}(z), [(C_2 \cup_{\gamma_2^{-1} \circ \gamma_{(1,1)}} C_1)^\circ] \rangle > .$$

This transition map  $\gamma_2^{-1} \circ \gamma_{(1,1)}$  is quite mysterious. We show here that it can be deformed to a transition map  $\rho$  which, while still not fully understood, has the property that it respects the fibration of the caps down to  $\mathbb{C}^N/\Gamma_\alpha \times X$ . With only this information and our work on the equivariant cohomology of  $\partial L$ , we are able to show the error term vanishes.

**Proposition 3.4.1.** *Let  $\rho_x$  be a family of embeddings parametrized by  $x \in X$*

$$\rho_x : (S^3 \times c(SO(3)) \times c(SO(3))) / \mathbb{Z}/(2) \rightarrow Z_2^s(\epsilon),$$

which are  $SO(4)$  and  $SO(3)$  equivariant with respect to the rotation and framing actions on the image and the action

$$[q, (A_1, t_1), (A_2, t_2)] \rightarrow [p_L q \bar{p}_R, (r A_1 \bar{p}_R, t_1), (r A_2 \bar{p}_R, t_2)],$$

on  $(S^3 \times c(SO(3)) \times c(SO(3))) / \mathbb{Z}/(2)$  for  $[p_L, p_R] \in SO(4), r \in SO(3), q \in S^3, [A_i, t_i] \in c(SO(3))$ . If we also assume that  $\rho_x$  sends  $S^3 \times c \times c$  to the trivial strata in  $Z_2^s(\epsilon)$  and we glue together  $C_1$  and  $C_2$  in a fiberwise manner to get the space

$$C_2 \cup_\rho C_1 = \mathbb{C}^N \times_{\Gamma_\alpha} (Q_\alpha \times_X Fr(X) \times_{SO(4)} (C_2^s(x) \cup_{\rho_x} C_1^s(x))) / S^1,$$

then we can define a link of the reducibles in  $C_2 \cup_\rho C_1$ , which we write as  $(C_1 \cup_\rho C_2)^\circ$ . We can then calculate that

$$\langle \bar{\mu}(z), [(C_2 \cup_\rho C_1)^\circ] \rangle > = 0.$$

*Proof.* This follows from Lemma 3.3.14 and the computation of  $(p_1)_*(\varphi^r)$  in Section 3.3. Arguing as in Section 3.2, we reduce the computation to an equivariant computation on  $ESO(4) \times_{SO(4)} (C_2^s(x) \cup_{\rho_x} C_1^s(x))^\circ / SO(3)$  of pushforwards of  $\varphi^r$ , where now,  $\varphi$  is Pontrjagin class of the  $SO(3)$  action here. We can give branched covers of  $C_i^s(x) / SO(3)$  by  $D^4 \times S^4$ . Again, the odd dimensional equivariant cohomology of the intersection of the two caps vanishes so we have unique lifts of  $\varphi$  to fiberwise compactly supported equivariant cohomology and the exact same computations give our result.

In the remainder of this section, we describe such a transition map  $\rho$  and then show that  $\gamma_2 \circ \gamma_{(1,1)}^{-1}$  can be deformed to  $\rho$ . If  $\gamma_2 \circ \gamma_{(1,1)}^{-1}$  can be deformed to  $\rho$ , we have  $C_2 \cup_{\gamma_2^{-1} \circ \gamma_{(1,1)}} C_1 \simeq C_2 \cup_\rho C_1$  and Proposition 3.4.1 will give the vanishing result.

The first approximation to our artificial transition map is constructed in [KoM]. The domain is a neighborhood of the boundary of the cap  $C_1$ . Consider the gluing data  $(A_0, [\phi_x, F_x, y, (A_1, t_1), (A_2, t_2)]) \in C_1$ , where  $A_0$  is a connection on  $Q_\alpha \times_{S^1} SO(3)$  which is anti-self-dual with respect to a metric  $g_t, \phi_x \in Q_\alpha|_x, F_x \in Fr(X)|_x, y \in D^4, x \in X$  and  $(A_i, t_i)$  are framed, charge one instantons on  $S^4$ . First, we compose the splicing map  $\gamma'_{(1,1)}$  with the map  $k$ , this gives an almost anti-self dual connection on  $X$ . The frame  $F_x$  identifies

a ball around  $x$ ,  $D_x^4$  with  $\mathbb{R}^4$ . Parallel radial translation (with respect to the connection  $A_0$ ) of  $\phi_x$  gives a trivialization of  $Q_\alpha \times_{S^1} SO(3)$  over  $B_x^4$ . If we write  $P$  for the  $SO(3)$  bundle obtained by gluing in the instantons, this trivialization of  $Q_\alpha \times_{S^1} SO(3)$  over  $D_x^4$  gives one of  $P$  over an annulus  $A_x$  whose outer boundary is equal to that of  $B_x^4$ . This trivialization makes  $\gamma'_{(1,1)} \circ k$  a connection on  $\mathbb{R}^4$ , framed at infinity. Cutting off this connection, we can extend it to a connection  $S^4$ . Keeping the radially parallel trivialization from the  $Q_\alpha$  frame gives a framed connection on  $S^4$ . This gives a map we call  $\rho'$ . We see  $\rho'$  is  $SO(4)$  equivariant with respect to the rotation action and the  $SO(3)$  action. The image of  $\rho'$  is an almost anti-self dual connection on  $S^4$ , so it can be perturbed to an anti-self dual one and then translated so it is centered. Both perturbation and translation commute with the rotation action so this operation is  $SO(4)$  and  $SO(3)$  equivariant on  $C_1^s(x)$ . Call this map  $\rho(A_0, x)$ . Let  $\rho_1$  be the extension of  $\rho(A_0, x)$  to  $\partial C_1$ .

**Remark** There is a deformation given in [KoM] between these transition maps, but the author is uncomfortable with their argument. They assert that the two gluing maps  $\gamma_{(1,1)}$  and  $\gamma_2 \circ \rho$  are close enough to imply the existence of a deformation between them. The best known bounds on the size of the perturbation map are  $L_1^2$  bounds in terms of the dilation parameter of the  $S^4$  connection being glued in. Better bounds (i.e.  $L_k^2$ ) would not seem possible as the cut-off function in the splicing construction would not allow better bounds on  $F_A^+$  than  $L^2$ . The images of the maps  $\gamma_{(1,1)}$  and  $\gamma_2 \circ \rho$  can be made close in the  $L_1^2$  metric topology, but only by restricting their domains to more concentrated connections. In the  $L^2$  metric topology, the ends of the moduli space look like cones and we would be making the two different embeddings close by pushing them down to the cone point. It seems reasonable that the  $L_1^2$  metric topology does not see this cone structure (the  $L_1^2$  metric is proportional to the conformally invariant  $L^4$  metric on the  $S^4$  moduli space), but there is a great deal to be checked. An explicit deformation between the two maps is presented here instead.

**Proposition 3.4.2.** *There is a deformation through embeddings between the transition maps  $\rho_1$  and  $\gamma_2^{-1} \circ \gamma_{(1,1)} \circ k$ .*

*Proof.* Our goal is to show that the map  $\gamma_{(1,1)} \circ k$  can be deformed, through embeddings, to  $\gamma_2 \circ \rho_1$ . We begin with the observation that the splicing map  $\gamma'_{(1,1)} \circ k$  is equal to the splicing map  $\gamma'_2 \circ \rho'$ . There are two ways to perturb the images of these two spaces onto the moduli space, arising from the two ways of creating right inverses to the linearization of the anti-self dual equation. Recall from [DK], Section 7.2 that a right inverse to  $d_{A'}^+$ , where  $A'$  is in the image of  $\gamma'_{(1,1)} \circ k$  can be constructed from splicing together right inverses from the background connection and the  $S^4$  instantons. The splicing construction depends on the frames and cut-off functions being used and thus will be different from the  $\Sigma_{(1,1)}$  and  $\Sigma_2$  construction. Note that although the  $S^4$  connection in the image of  $\rho'$  is not anti-self-dual, it is sufficiently close to the anti-self-dual connections that this construction can still be applied to it. We thus have two different right inverses,  $P_{(1,1)}$  and  $P_2$  to  $d_{A'}^+$ , and thus two different perturbations of the image of  $\gamma'_{(1,1)} \circ k$  to the moduli space. We construct a family of right inverses  $P_t = (1-t)P_{(1,1)} + tP_2$  to  $d_{A'}^+$ , which will satisfy the necessary bounds (e.g. Lemmas 7.2.18 and 7.2.23 in [DK]) and thus define a family of perturbations of the splicing map to the moduli space. Our first deformation of  $\gamma_{(1,1)} \circ k$  is to change the perturbation of  $\gamma'_{(1,1)} \circ k$  to the moduli space by using this family of right inverses. Thus we have deformed the map  $\gamma_{(1,1)} \circ k$  to the map given by composing  $\gamma'_2 \circ \rho'$  with the perturbation to the moduli space defined by a right inverse constructed from the gluing data of the lower stratum. As noted above, although the image of  $\rho'$  is not anti-self dual, it is close enough to the anti-self dual connections to have a right inverse with the appropriate bounds and the map  $\gamma_2 \circ \rho'$  can be defined by adding a perturbation of  $\gamma'_2 \circ \rho'$  to make the image anti-self dual. Now, the map  $\rho_1$  is a deformation of  $\rho'$  (the perturbation is a deformation). We deform  $\gamma'_2 \circ \rho'$  to  $\gamma'_2 \circ \rho_1$  and thus deform the maps to the moduli space  $\gamma_2 \circ \rho'$  to  $\gamma_s \circ \rho_1$ . We end up with the map  $\gamma_2 \circ \rho_1$ .

Our new transition function  $\rho_1$  does not yet satisfy the conditions of Proposition 3.4.1 because it depends

on the background connection  $A_0$  and the metric  $g_t$ . But our family of background connections and metrics is contractible, so the map  $\rho_1$  can be deformed to  $\rho_1([\alpha], x)$ , where  $\alpha$  is the reducible connection.

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